THE THEORY, ERROR ANALYSIS AND PRACTICAL PROBLEMS ENCOUNTERED IN THE DESIGN OF NON-INTERACTING CONTROL SYSTEMS

by

R. T. Darcy

and

D. K. Forbes

May 1958

Preface

This study was conducted at the University of Michigan for partial fulfillment of the requirements for the degree of Master of Science in Engineering, during the Fall and Spring semesters of 1957-1958.

The authors wish to acknowledge their indebtedness to Dr. Edward O. Gilbert for his invaluable guidance and assistance in this undertaking.

Contents

Chapter I

Introduction

Problems frequently encountered in complex multiple input-ouput systems stem from interaction or coupling between the controlled variables. For example, in the lateral control of an aircraft, a change in the angle of bank by aileron deflection will result in a change in yaw angle provided that there is no rudder input to offset the change. To eliminate this interaction between the controlled outputs, the theory of non-interacting controls was developed. For a control system designed in accordance with this theory, a desired change in one output variable will not cause a change in the other output variables.

The general method for designing such non-interacting control systems was first developed by A. S. Boksenbom and R. Hood, Ref. 1, and applied by them to the control of a turbo-jet engine. This work was later repeated by H. S. Tsien, in Ref. 2. In Ref. 3, H. Freeman extended the theory to obtain a desired degree of interaction in an m input and n output system. In Ref. 4, Freeman discussed stability and realizability but he did not discuss the practical design problems or use the more realistic open loop approach.

This study will develop the basic criteria for non-interaction of the variables in an n input and n output system. In addition, the effect of disturbance inputs and errors in mathematically describing the physical system and synthesizing the controller will be discussed. It will also introduce the potential designer of a non-interacting control system to some of the practical problems which are often encountered in applying the basic non-interaction

criteria. As a final step, the theory will be applied to the design of an autopilot system in order to illustrate the design techniques and the difficulties that may be encountered in a practical application.

Chapter II

General Theory

In this chapter, the basic relations between controller and controlled elements of a system to effect non-interaction of variables will be developed. To achieve this, a specific system configuration will be assumed. However, the extension of this theory to other configurations is feasible and should be apparent from the subsequent development.

Introduction

A linear physical system with n inputs, $\bar{\delta}_i$, and n outputs, $\bar{y}_i$, can be simply described in vector notation. In such notation, $\bar{\delta}$ and $\bar{y}$ are considered as n dimensional vectors with components $\bar{\delta}_i$ and $\bar{y}_i$. The coefficients of the equations relating $\bar{y}$ to $\bar{\delta}$ in the physical system form an n by n matrix, $\bar{A}$, which is composed of elements $\bar{a}_{ij}$. Note that the $\bar{a}_{ij}$ are operators expressable as $\bar{a}_{ij}(p)$ where $p = d/dt$. Thus,

$$\bar{y}_i = \sum_{j=1}^{n} \bar{a}_{ij} \bar{\delta}_j \qquad \text{for } i = 1 \text{ to } n \tag{2-1}$$

or in vector-matrix notation,

$$\bar{y} = \bar{A} \bar{\delta} \tag{2-2}$$

In a block diagram, the system described above can be represented as:

$$\bar{\delta} \longrightarrow \boxed{\;\bar{A}\;} \longrightarrow \bar{y}$$

In the case where there are m inputs and n outputs and m is greater than n, the m-n additional inputs are fed directly through the system to become system

outputs. This implies that the A matrix is now an m x m matrix with the added $\bar{a}_{ii}$ elements having a value of unity. Such a case is described at length in Refs. 1 and 2.

To achieve non-interaction, it is desired that $\bar{y} = \bar{x}$ where X is an n dimensional input vector with components x_i. In designing a linear system to satisfy the above input-output requirements, the foremost problem is to use the error vector, $\epsilon = \bar{y} - X$, to determine $\bar{\delta}$ for non-interaction and at the same time ensure that $\epsilon \approx 0$. One possible system configuration for achieving these ends is shown in Fig. 1(a).

In order to keep the treatment of the subject sufficiently general, servos, transducers and controllers have been included in the overall system shown in Fig. 1(a). The transducers have been supposed in light of the fact that the real outputs, $\bar{y}_i$, of a physical system can not be obtained directly. Thus the measured outputs, y_i, are related to the real outputs, $\bar{y}_i$, through the transducer matrix, T. This matrix is a diagonal matrix with elements t_{ii}. It is further supposed that the transducers are selected such that the $\bar{y}_i - y_i$ are very small and hence no attempt will be made to correct y_i to $\bar{y}_i$. The physical system inputs, $\bar{\delta}_i$, are generally servo driven and the $\bar{\delta}_i$ are related to the servo signal inputs, δ_i, through the servo matrix, S. Like the T matrix, S is a diagonal matrix with elements s_{ii}. The controller, which relates the system input signal vector, δ, to the error signal vector, ϵ, is represented by an n by n matrix, C. It will be designed such that the overall system is non-interacting and has desired response characteristics.

As a means of simplifying the ensuing mathematical analysis, the system

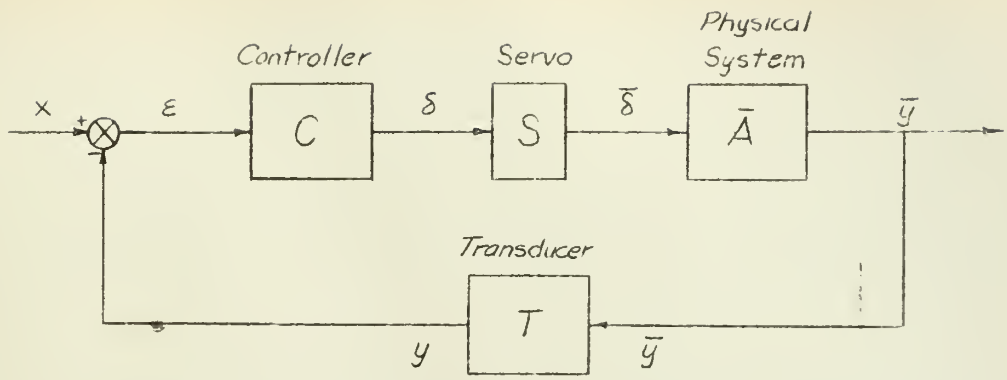


Fig. 1(a)

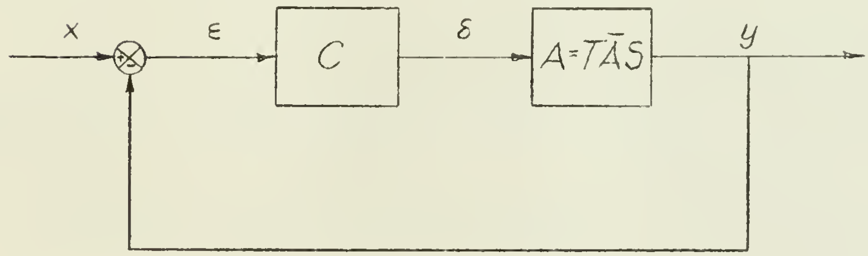


Fig. 1(b)

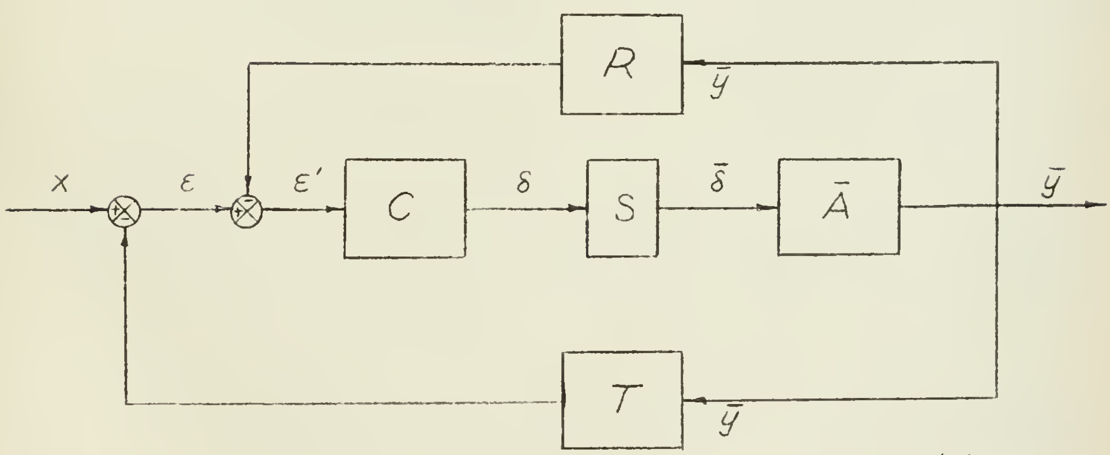


Fig. 1(c)

Fig. 1 Typical Control System Configurations

shown in Fig. 1(b) shall be used. It should be noted that this is an equivalent system to that of Fig. 1(a) and that many multiple input-output systems can be put into this form. This does not mean that this is the only form possible. However, conditions for non-interaction are more complex and system design becomes exceedingly difficult, if not impossible, when configurations different from Figs. 1(a) and 1(b) are used. A simple exception to the above is shown in Fig. 1(c). Here the diagonal matrix R, which might for example provide derivative feedbacks, has been added. Although another loop has been added, the system of Fig. 1(c) reduces to that of Fig. 1(b) if A = $(T + R)\bar{A}S$ and $y = (T+R)\bar{y}$.

Analysis of the Control System

The system under consideration is the simplified but equivalent system shown in Fig. 1(b). The vector equation for this system is:

$$y = A\delta = AC\epsilon = AC(x-y) \tag{2-3}$$

Solving equation (2-3) for the system output vector, y , provides:

$$y = \left[I + AC\right]^{-1} AC\,x \tag{2-4}$$

where I is the identity matrix.

In order that the inverse matrix $\left[I+AC\right]^{-1}$ shall exist, it is required that the determinant, $\left|I+AC\right|$, is not identically zero over all values of the complex variable p. Of course, AC will be a function of the operator p and hence $\left|I+AC\right|$ may be equal to zero for a few particular values of the operator p considered as a complex variable.

If the system under consideration is to be non-interacting it is required that:

$$y = \bar{D}x \tag{2-5}$$

where $\bar{D}$ is a diagonal matrix with elements $\bar{d}_{ii}$. This also implies that $y_i = \bar{d}_{ii} x_i$ for all i or that a specific output is affected only by a corresponding input.

From equations (2-4) and (2-5) it follows that:

$$[I+AC]^{-1}AC = \bar{D} \tag{2-6}$$

Certainly, one condition on the matrix AC that satisfies the above equation is that AC is a diagonal matrix. If AC is diagonal then $[I+AC]^{-1}AC$ is given by:

$$[I+AC]^{-1}AC = \begin{bmatrix} \frac{(ac)_{11}}{1+(ac)_{11}} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \frac{(ac)_{22}}{1+(ac)_{22}} & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & & \cdot & & & & & \cdot \\ \cdot & 0 & & \cdot & & & & 0 \\ \cdot & \cdot & & & \cdot & & & \\ \cdot & \cdot & & & & & & \frac{(ac)_{ii}}{1+(ac)_{ii}} \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & & \end{bmatrix} \tag{2-7}$$

where $(ac)_{ii}$ are the components of AC.

Solving equation (2-6) for AC yields:

$$AC = [I-\bar{D}]^{-1}\bar{D} \tag{2-8}$$

Again, it is obvious that AC is a diagonal matrix if equation (2-8) is to be satisfied.

In summarizing, a necessary and sufficient condition for non-interaction is that the matrix AC be diagonal; i.e. AC = D. Furthermore, the elements

$(ac)_{ii}$ or d_{ii} must be finite and not identically minus one. This assures that:

$$y_i = \bar{d}_{ii} x_i = \frac{(ac)_{ii}}{1+(ac)_{ii}} x_i = \frac{d_{ii}}{1+d_{ii}} \quad \text{for } i = 1 \text{ to } n \tag{2-9}$$

where $\bar{d}_{ii}$ is finite.

It is now known that AC = D where D is a diagonal matrix different from the diagonal matrix $\bar{D}$. Solving for C by matrix methods yields:

$$C = A^{-1}D \tag{2-10}$$

For A^{-1} to exist, the determinant of A must not be identically equal to zero. This requirement is also necessary for n inputs to specify n outputs independently. Thus, a condition where $|A| \equiv 0$ implies a lack of independence among the system equations and the impossibility of completely controlling n outputs y_i by n inputs δ_i.

Equation (2-10) can be rewritten as:

$$c_{ij} = \sum_{k=1}^{n} a^{-1}_{ik} d_{kj} \tag{2-11}$$

D is a diagonal matrix with elements d_{ii} and therefore the terms in the summation are non-zero only for k = j. Thus,

$$c_{ij} = a^{-1}_{ij} d_{jj} \tag{2-12}$$

Considering the case where j = i, equation (2-11) reduces to $c_{ii} = a^{-1}_{ii} d_{ii}$ or:

$$d_{ii} = \frac{c_{ii}}{a^{-1}_{ii}} \tag{2-13}$$

Note that equation (2-13) is the open loop transfer function of the overall system.

Here a^{-1}_{ii} is known and c_{ii} is designed so that the closed loop transfer function, $\frac{d_{ii}}{1+d_{ii}}$, provides favorable response characteristics.

After the c_{ii} have been found by conventional compensation methods, the off diagonal elements of the controller matrix, c_{ij}, are found by combining equations (2-12) and (2-13) to obtain:

$$c_{ij} = \frac{a^{-1}_{ij} c_{jj}}{a^{-1}_{jj}} \qquad \text{for} \begin{cases} i = 1 \text{ to } n \\ j = 1 \text{ to } n \end{cases} \quad i \neq j \tag{2-14}$$

If any of the $a^{-1}_{jj} \equiv 0$, then the choice of d_{jj} is arbitrary. It follows that the off diagonal components, c_{ij}, are then determined from equation (2-12). Although d_{jj} is arbitrary when $a^{-1}_{jj} \equiv 0$, it should be chosen such that good closed loop response characteristics are obtained. Thus, it should be chosen fairly large in order that outputs approximate inputs with a minimum steady state error. This may be seen from the relation, $y_i = \frac{d_{ii}}{1+d_{ii}} x_i$. Naturally, the d_{ii} can not be overly large otherwise the c_{ij} must assume impractical proportions through equation (2-12).

In summary, the c_{ii} are determined by the design of the open loop transfer function, $d_{ii} = \frac{c_{ii}}{a^{-1}_{ii}}$ where a^{-1}_{ii} is known. Thus the c_{ii} are essentially compensation so chosen that $\frac{d_{ii}}{1+d_{ii}}$ is stable with favorable response characteristics. The off diagonal elements of the controller matrix are then found from equation (2-14). Provided no difficulties are encountered in compensating the d_{ii}, the system performance should be as desired, namely, non-interaction of control variables.

Chapter III

Error Analysis

There are essentially two basic types of system errors. The first type of error is a dynamic error resulting from random disturbances entering the system. The second type are those errors introduced into the system through the A and C matrices due to inaccuracies in describing the true physical system and/or inaccuracies in the synthesis of the controller elements or transfer functions. Errors of the second type could obviously result in some system interaction and conceivably stability problems.

Disturbance Errors

In analyzing errors of the first type, the disturbances were considered as input signals to the A and C matrices as shown in Fig. 2. It is supposed that m disturbance signals, $\mathcal{U}_\alpha$, are introduced into the C matrix and r disturbance signals, $\mathcal{V}_\beta$, are introduced into the A matrix. These input signals may be represented as an m dimensional vector, $\mathcal{U}$, and an r dimensional vector, $\mathcal{V}$. As an example of the source of these inputs, the $\mathcal{U}$ type of input might be considered as n^2 noise or drift type signals introduced at the inputs of the n^2 c_{ij} components. The $\mathcal{V}$ type of signals are introduced into the physical system being controlled and hence could be torque loads, gust effects, etc. Regardless of their source, it is desirable to minimize the effect of the $\mathcal{U}$ and $\mathcal{V}$ disturbances.

It will be assumed that $\mathcal{U}$ and $\mathcal{V}$ are related to the system in vector-matrix form as follows:

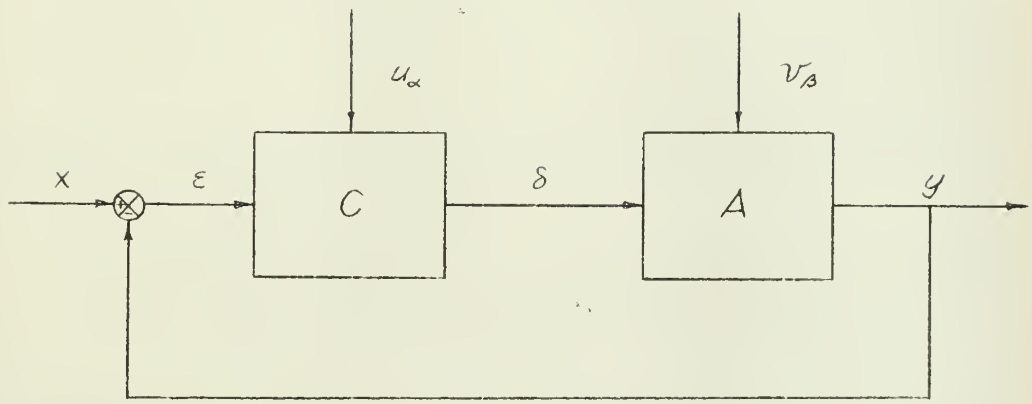


Fig. 2 Introduction of Disturbances into the Control System

$$\delta = C\epsilon + C^{*}u$$
$$y = A\delta + A^{*}V \qquad (3\text{-}1)$$

In the above equations, C^* is an n by m matrix and A^* is an n by r matrix. Noting that $\epsilon = x - y$, the system error equation can be found from equations (3-1) to be:

$$\epsilon = +[I + AC]^{-1}(x - AC^{*}u - A^{*}V) \qquad (3\text{-}2)$$

Specifically, the errors due to the u and V inputs can be expressed as:

$$\epsilon_u = -[I + AC]^{-1} AC^{*}u$$
$$\epsilon_v = -[I + AC]^{-1} A^{*}V \qquad (3\text{-}3)$$

In designing a control system, the designer will have to investigate the errors resulting from the u and V disturbances. This will require an investigation of the above vector equations to determine the error associated with the specific input disturbance. This may be illustrated by considering the effect of a disturbance input to a single component of the C matrix in a two-dimensional system. Assuming the disturbance as an input, u_1, to the c_{12} component of the C matrix, equation (3-3) becomes:

$$\epsilon_{u_1} = -[I + AC]^{-1} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} c_{12} \\ 0 \end{bmatrix} u_1$$

or

$$\epsilon_{u_1} = -[I + AC]^{-1} \begin{bmatrix} (a_{11} c_{12}) \\ (a_{21} c_{12}) \end{bmatrix} u_1$$

As developed previously, for non-interaction, AC is a diagonal matrix which was denoted as D. Therefore,

$$\epsilon_{u_1} = -\begin{bmatrix} \frac{1}{1+d_{11}} & 0 \\ 0 & \frac{1}{1+d_{22}} \end{bmatrix} \begin{bmatrix} (a_{11}C_{12}) \\ (a_{21}C_{12}) \end{bmatrix} u_1$$

and

$$\epsilon_{1u_1} = -\frac{a_{11}C_{12}}{1+d_{11}}u_1$$

$$\epsilon_{2u_2} = -\frac{a_{21}C_{12}}{1+d_{22}}u_1$$

For illustration purposes, ϵ_{1u_1} will be the only error considered. From equations (2-13) and (2-14):

$$d_{11} = \frac{C_{11}}{a^{-1}_{11}} \quad ; \quad C_{12} = \frac{a^{-1}_{12}}{a^{-1}_{22}}C_{22} = -\frac{a_{12}}{a_{11}}C_{22}$$

$$C_{21} = \frac{a^{-1}_{21}}{a^{-1}_{11}}C_{11} = -\frac{a_{21}}{a_{22}}C_{11}$$

Therefore,

$$\epsilon_{1u_1} = \frac{a_{12}C_{22}}{1 + C_{11}/a^{-1}_{11}}$$

Assuming the hypothetical system represented by:

$$A = \begin{bmatrix} \frac{1}{p} & \frac{.2}{p(p+1)} \\ \frac{.2}{p+1} & \frac{1}{p+1} \end{bmatrix}$$

then,

$$\frac{1}{a^{-1}_{11}} = \frac{|A|}{a_{22}} = \frac{p+.96}{p(p+1)} \cong \frac{1}{p} \; ; \; \frac{1}{a^{-1}_{22}} = \frac{p(p+.96)}{p(p+1)^2} \cong \frac{1}{p+1}$$

Now, if the control matrix diagonal elements are designed as:

$$C_{11} = 1 \qquad\qquad C_{22} = \frac{1}{p}$$

then,

$$C_{12} = -.2\left[\frac{1}{p(p+1)}\right] \quad ; \quad C_{21} = -.2$$

Therefore,
$$\epsilon_{1u_1} = \frac{.2}{p}\left[\frac{1}{(p+1)^2}\right]u_1$$

Since $\epsilon = x - y$, then the output y_1, with x_1 assumed as zero, will be:

$$y_1 = -\frac{.2}{p}\left[\frac{1}{(p+1)^2}\right]u_1$$

It is readily apparent that this transfer function presents an undesirable situation, since at low frequencies the magnitude of the output will be large because of the term .2/p. In addition, the output diverges with time for a step input, u_1. Thus, in this hypothetical case the designer would be forced to modify the controller elements in order to provide good system response and at the same time minimize the disturbance errors, which may be difficult, if not impossible.

One method of reducing the effect of the disturbances which may prove effective in a practical design is to introduce a high gain prior to the disturbance inputs. Fig. 3(a) illustrates a basic circuit for a scalar system (one of the non-interacting subsystems for example) where, with $x=0$, the transfer function relating the output, y, to the disturbance input, u, is:

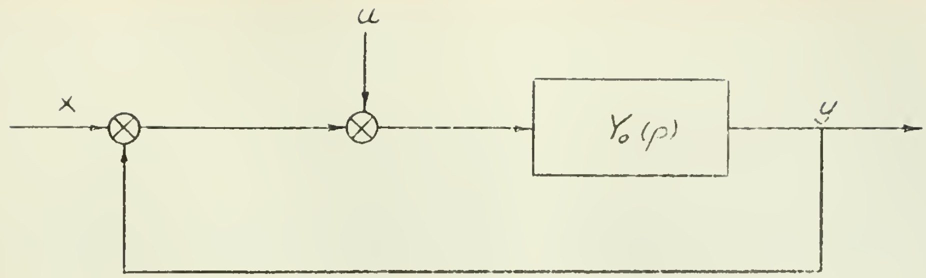


Fig. 3(a)

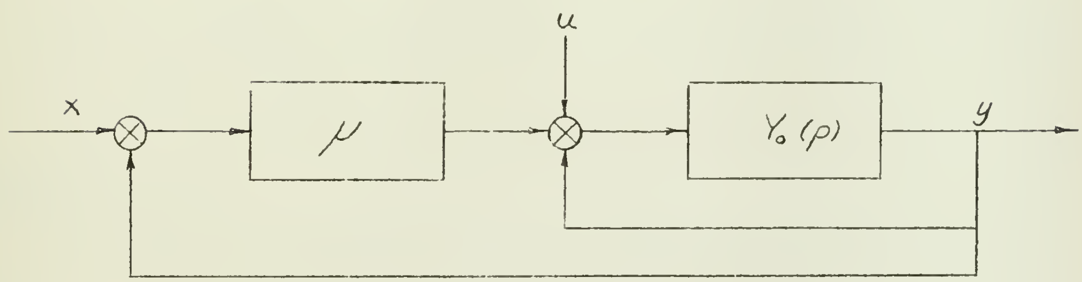


Fig. 3(b)

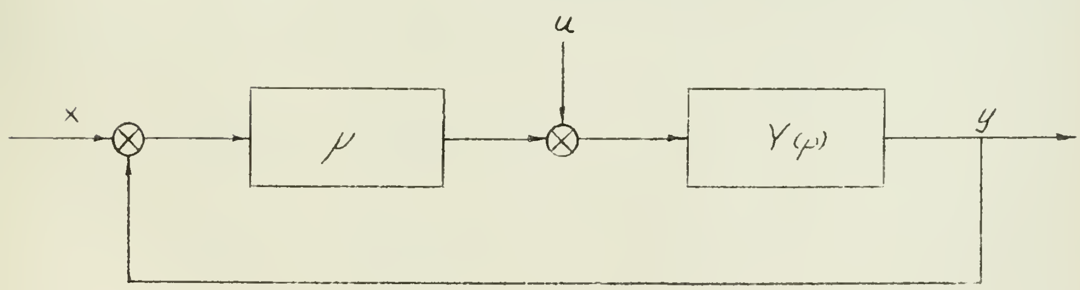


Fig. 3(c)

Fig. 3 A Method for Reducing the Effect of Disturbances Entering the Control System

$$\frac{y}{u} = \frac{Y_0(p)}{1+Y_0(p)} = Y(p) \tag{3-4}$$

With the introduction of a high gain element μ as in Fig. 3(b), the circuit can be reduced to that shown in Fig. 3(c). From this latter figure, with $X = 0$,

$$\frac{y}{u} = \frac{Y(p)}{1+Y(p)\mu} \tag{3-5}$$

By proper design, μ can introduce high gain over the desired frequency range and consequently reduce the output due to the disturbance inputs. μ is, of course, designed to maintain good system characteristics for y with X as an input.

Component Errors

To analyze the second type of error it will be necessary to refer to equation (3-2) which is repeated below:

$$\epsilon = [I + AC]^{-1}(X - AC^*u - A^*V) \tag{3-2}$$

The elements of the A, A*, C and C* matrices can be redefined in terms of a lowest common denominator as follows:

$$\begin{aligned} a_{ij} &= \frac{\tilde{a}_{ij}(p)}{d_a(p)} \\ a^*_{ij} &= \frac{\tilde{a}^*_{ij}(p)}{d_a(p)} \\ c_{ij} &= \frac{\tilde{c}_{ij}(p)}{d_c(p)} \\ c^*_{ij} &= \frac{\tilde{c}^*_{ij}(p)}{d_c(p)} \end{aligned} \tag{3-6}$$

where $\tilde{a}_{ij}$, $\tilde{C}_{ij}$, $\tilde{a}^*_{ij}$, $\tilde{C}^*_{ij}$ are polynomials in the operator p and $d_a(p)$ and $d_c(p)$ are the lowest common denominators of the a_{ij} and c_{ij} components respectively.

From the above definitions, it is apparent that the a^*_{ij} have the same poles as the a_{ij} by definition. The same is true for c^*_{ij} and c_{ij}. This generally is the case and if it is not, then the nonacceptable poles may be removed and used to define a new vector, $\mathcal{U}$ or $\mathcal{V}$. It is now possible to define new matrices whose elements are polynomials in the operator p. Thus,

$$\tilde{A} = (\tilde{a}_{ij}) \quad ; \quad \tilde{A}^* = (\tilde{a}^*_{ij}) \qquad\qquad (3\text{-}7)$$
$$\tilde{C} = (\tilde{C}_{ij}) \quad ; \quad \tilde{C}^* = (\tilde{C}^*_{ij})$$

Note that $A = \frac{1}{d_a}\tilde{A}$, $A^* = \frac{1}{d_a}\tilde{A}^*$, $C = \frac{1}{d_c}\tilde{C}$, $C^* = \frac{1}{d_c}\tilde{C}^*$ Hence, equation (3-2) may be written as:

$$\epsilon = \left[d_a d_c I + \tilde{A}\tilde{C}\right]^{-1}\left(d_a d_c X - \tilde{A}\tilde{C}^*\mathcal{U} - d_c\tilde{A}^* V\right) \qquad (3\text{-}8)$$

In the above equation, the components of $\tilde{A}$, $\tilde{A}^*$, $\tilde{C}$, $\tilde{C}^*$, d_a and d_c are polynomials in the operator p. Hence any poles of equation (3-8) result from the determinant of $\left|d_a d_c I + \tilde{A}\tilde{C}\right|$. If determinant $\left|d_a d_c I + \tilde{A}\tilde{C}\right|$ has no zeroes in the right half p plane, or on the jw axis (imaginary axis), then the system is stable for disturbance inputs as well as normal inputs, X_i. However, if determinant $\left|d_a d_c I + \tilde{A}\tilde{C}\right|$ and $d_a d_c$ have zeroes which are common, and if these zeroes are in the right half p plane then the system will be stable for X inputs but could be unstable for $\mathcal{U}$ and V inputs. If $d_a d_c$ has no zeroes in the right half p plane and the system is designed so that

$\frac{d_{11}}{1+d_{11}}$ and $\frac{d_{22}}{1+d_{22}}$ are stable, then the error vector, ϵ , must be stable for all inputs, X , $\mathcal{U}$ and V .

When there are small changes in the coefficients of A and C due to inaccuracies in the synthesis of C or inexact knowledge of the coefficients of the equations of the physical system, the zeroes of the determinant of $\left| d_a d_c I + \tilde{A}\tilde{C} \right|$ will change because of the coefficient change. Hence, the poles of equation (3-8) will be altered slightly for small changes in the coefficients. Thus, if the system is designed stable and if $d_a d_c$ has no zeroes in the right half complex plane we are assured that the system remains stable.

The small changes in the coefficients of A and C will result in interaction. However, if the deviations are truly small, then the resulting interaction will be negligible. This is due to the fact that the d_{ij} or off diagonal elements will be small and hence will have little effect on the overall system. Thus, it may be seen that interaction due to these slight deviations can be minimized only through careful attention to the analytical description of the physical system and accurate design and mechanization of the controller.

Chapter IV

Practical System Problems

In the previous chapter, the effect of small deviations, ϵ , in a_{ij} and c_{ij} were discussed. In this chapter, the problems or consequences attendant with gross deviations in c_{ij} because of design advantages will be investigated. In addition, problems arising from impossible or impractical compensation requirements, as may be dictated by non-interaction theory, will be discussed and means of alleviating the problem pointed out.

Problems arising from gross deviations in c_{ij}.

Suppose that in synthesizing complicated controller transfer functions, c_{ij}, simplifications are made by neglecting terms in the numerator and denominator of c_{ij} that have "break" frequencies well above the natural frequency of the system. Further suppose that all these discarded terms have negative roots. The obvious consequence of such gross deviations from the desired c_{ij} is system interaction at the higher frequencies. In addition, one must consider the effect of these departures on the stability of the overall system.

To investigate these ideas more fully, assume a two-dimensional control system in which the exact controller transfer functions, c_{12} and c_{21} are altered by a factor q to produce new relations c'_{12} and c'_{21}. Analytically,

$$C'_{12} = q_1 C_{12} \tag{4-1}$$

$$C'_{21} = q_2 C_{21} \tag{4-2}$$

For the theoretical non-interacting system, the following equations have been shown to exist:

$$d_{11} = a_{11} C_{11} + a_{12} C_{21} \equiv Y_{0_1} = C_{11}/a_{11}^{-1} \quad (4\text{-}3)$$

$$d_{22} = a_{21} C_{12} + a_{22} C_{22} \equiv Y_{0_2} = C_{22}/a_{22}^{-1} \quad (4\text{-}4)$$

$$d_{12} = a_{11} C_{12} + a_{12} C_{22} = 0 \quad (4\text{-}5)$$

$$d_{21} = a_{21} C_{11} + a_{22} C_{21} = 0 \quad (4\text{-}6)$$

Substituting equations (4-1) and (4-2) into equations (4-3) through (4-6) where applicable and manipulating and rearranging where desired, the results are:

$$d_{11}' = a_{11} C_{11} + a_{12} q_2 C_{21} = \left[a_{11} + q_2 a_{12}\, a_{21}^{-1}/a_{11}^{-1}\right] C_{11} \quad (4\text{-}7)$$

$$d_{22}' = a_{22} C_{22} + q_1 a_{21} C_{12} = \left[a_{22} + q_1 a_{21}\, a_{12}^{-1}/a_{22}^{-1}\right] C_{22} \quad (4\text{-}8)$$

$$d_{12}' = a_{11} q_1 C_{12} + a_{12} C_{22} = \left[a_{12} (1-q_1) C_{22}\right] \neq 0 \quad (4\text{-}9)$$

$$d_{21}' = a_{21} C_{11} + q_2 C_{21} a_{22} = \left[a_{21} (1-q_2) C_{11}\right] \neq 0 \quad (4\text{-}10)$$

where the d'_{ij} are elements of a new and non-diagonal matrix, D'.

The interaction is evident from equations (4-9) and (4-10). The changes in the transfer functions d_{ii} due to the factors q_i may be determined from equations (4-7) and (4-8).

In investigating the effects of gross deviations on system stability, it is best to look at the system equation in vector-matrix notation. It is:

$$y = [I + D']^{-1} D' x$$

where D' is a non-diagonal matrix.

The stability of the closed loop system is determined from the determinant of $|I+D'|$ which is the characteristic equation of the overall system. It can be shown that:

$$|I+D'| = (1+d'_{11})(1+d'_{22}) - d'_{12}d'_{21} \tag{4-11}$$

It follows that if this resulting polynomial has zeroes in the right half complex plane the overall system is closed loop unstable.

One means of determining the stability of the closed loop system without solving for the roots of the polynomial is as follows:

$$(1+d'_{11})(1+d'_{22}) - d'_{12}d'_{21} = 0$$

or when rewritten:

$$(1+d'_{11})(1+d'_{22})\left[1 - \frac{d'_{12}d'_{21}}{(1+d'_{11})(1+d'_{22})}\right] = 0 \tag{4-12}$$

Stability is assured if none of the above factors have zeroes in the right half p plane.

The open loop transfer functions, d'_{11} and d'_{22}, which are given in equations (4-7) and (4-8), should be plotted in Nyquist plots and Nyquist stability criteria, N = P - Z applied to determine if there are zeroes of the $1 + d'_{ii}$ in the right half p plane.

Assuming that there are no zeroes of the $1 + d'_{ii}$ in the right half p plane and that d'_{12} and d'_{21} have no poles in the right half p plane, then for system stability, the absolute value of $\dfrac{d'_{12}\,d'_{21}}{(1+d'_{11})(1+d'_{22})}$ must be less than unity for all frequencies, ω . However, reference to the Nyquist plot of a typical d'_{ii} shown in Fig. 4 provides the following important relationship:

$$\left| 1 + d'_{ii} \right| \geq a \qquad \text{where} \qquad a = \frac{1}{M_i + 1} \tag{4-13}$$

It follows from equation (4-13) that:

$$\frac{d'_{12}\,d'_{21}}{\left(\frac{1}{1+M_1}\right)\left(\frac{1}{1+M_2}\right)} \geq \frac{d'_{12}\,d'_{21}}{(1+d'_{11})(1+d'_{22})}$$

Thus, if $(1 + M_1)\,(1 + M_2)\,d'_{12}\,d'_{21} < 1$ for all frequencies, ω , and $(1 + d'_{11})$ as well as $(1 + d'_{22})$ have no zeroes in the right half p plane, then the overall control system is closed loop stable.

Compensation Problems

It is conceivable that the co-factors of the a_{ii} of the physical system may have zeroes in the right half p plane so that the $\dfrac{1}{a'_{ii}}$ will have poles in the right half p plane. From the basic open loop non-interaction relation, $d_{ii} = \dfrac{c_{ii}}{a'_{ii}}$, it follows that it may be impossible or impractical to compensate $\dfrac{1}{a'_{ii}}$ with c_{ii} so that $\dfrac{d_{ii}}{1+d_{ii}}$ is stable. In this event, it would be impossible to obtain a practical non-interacting control system using the design techniques discussed in the theory section. As an example, consider the following simple system:

$$A = \begin{bmatrix} \dfrac{1}{p+1} & \dfrac{.2}{p+1} \\[2ex] \dfrac{.2}{p+1} & \dfrac{-.01p+1}{p+1} \end{bmatrix} \quad ; \quad \delta = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}$$

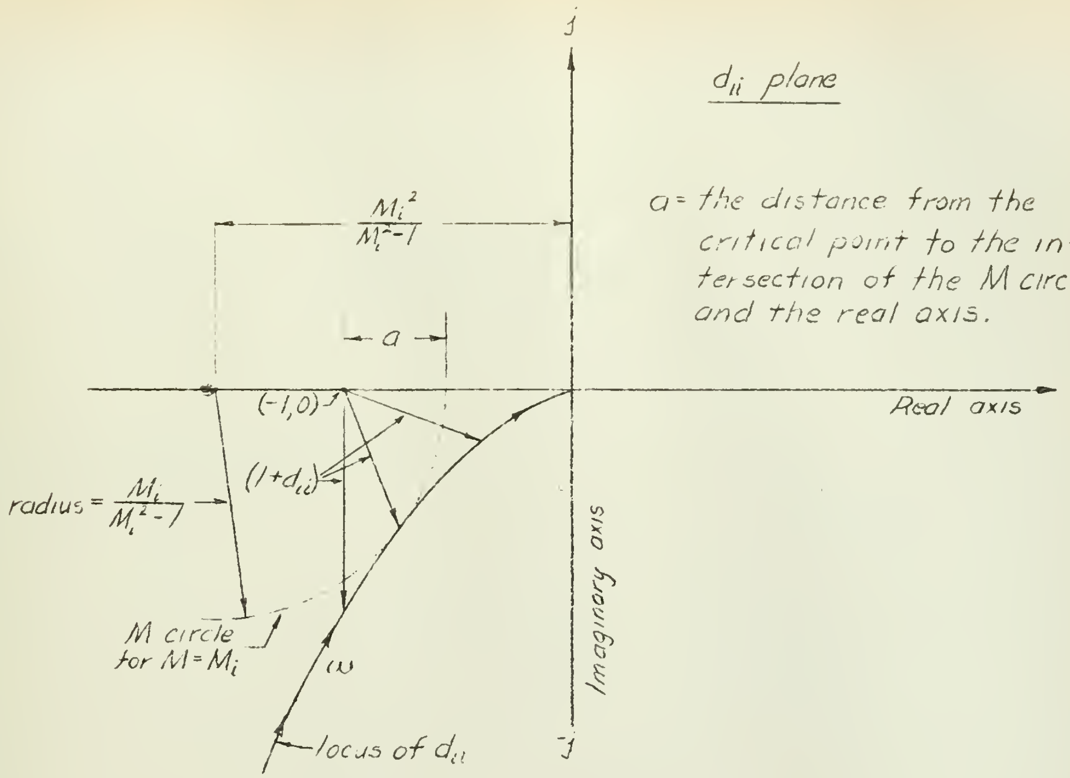
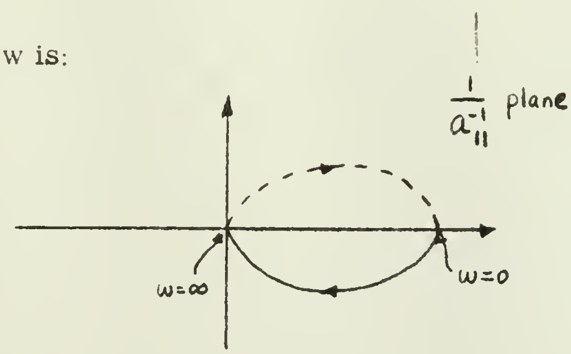


Fig. 4 Nyquist Plot of a Typical Open Loop Transfer Function, d_{ii}

Then: $$a_{11}^{-1} = \frac{(-.01p+1)(p+1)}{-.01p+.96}$$

and: $$\frac{1}{a_{11}^{-1}} = \frac{-.01p+.96}{(-.01p+1)(p+1)}$$

The locus of $\frac{1}{a_{11}^{-1}}$ for p = jw is:



But for stability, N = P - Z and in this case P = 1. Therefore, N must be + 1 if Z is to be zero. However, from the general non-interaction theory, $d_{11} = \frac{c_{11}}{a_{11}^{-1}}$. It is quite obvious that the compensation of $\frac{1}{a_{11}^{-1}}$ such that N = + 1 and Z = 0 is a difficult problem. One means of compensation is to use the prediction operator, e^p. This will provide the necessary encirclements but is entirely impractical as a means of compensation in a control problem. Other than this, no compensation could be found. One might consider mechanizing the compensation so that it cancelled the objectionable poles and zeroes. However, this will not work because the poles and zeroes of the physical system are not known exactly and unless we have exact cancellation the poles and zeroes are not really cancelled. Thus, it appears that the straight forward design techniques developed in the theory can not be used when confronted with such a compensation problem.

In some cases the foregoing design problem can be avoided by merely interchanging the rows of the δ vector. This implies an interchange of controls. If the design problem is removed by such a change, one may continue to design the control system in accordance with the general non-interaction theory.

As an example of the foregoing, consider the same system of the previous example but with the δ vector rows interchanged to form a vector δ' and a corresponding change of the columns of A to form a new matrix A'. Thus,

$$A' = \begin{bmatrix} \frac{.2}{p+1} & \frac{1}{p+1} \\ \frac{-.01p+1}{p+1} & \frac{.2}{p+1} \end{bmatrix} \quad ; \quad \delta' = \begin{pmatrix} \delta_2 \\ \delta_1 \end{pmatrix}$$

The new d_{ii} are now:

$$d_{11} = Y_{o_1} = \frac{C_{11}|A'|}{a'^{-1}_{11}} = \frac{-C_{11}(.2)(-.01p+.96)}{p+1}$$

$$d_{22} = Y_{o_2} = \frac{C_{22}|A'|}{a'^{-1}_{22}} = \frac{-C_{22}(.2)(-.01p+.96)}{p+1}$$

There are no poles in the right half p plane of the above open loop transfer functions. Hence, P = 0. In addition, the design of c_{11} and c_{22} so as to have N = Z = 0 in the closed loop system is not anticipated to present any real compensation difficulties. Thus, the impossible or impractical compensation problem has been avoided. It should be emphasized that interchanging the rows of the δ vector will not alleviate this particular compensation problem in all

instances. This is obvious from the foregoing example if either $\frac{1}{a^{-1}_{12}}$ or $\frac{1}{a^{-1}_{21}}$ have one or more poles in the right half p plane. Naturally, the presence of such poles does not imply that the $\frac{1}{a^{-1}_{ij}}$ can not be practically compensated so that Z = 0 but rather that difficulties in compensation are more probable than the instances when P = 0.

In a more complicated system where there are n control inputs and outputs and n is greater than two, the poles of $\frac{1}{a^{-1}_{ii}}$ result from the co-factors of the A matrix. These complicated co-factors tend to increase the probability that $\frac{1}{a^{-1}_{ii}}$ will have one or more poles in the right half p plane. Thus, one might expect some difficult and impossible compensation problems when applying the general non-interaction theory to complicated physical systems. If an interchange of the controls, (interchange δ_i) does not avoid these problems then a different design approach must be taken.

The theory behind one approach and the effect on system performance is most easily described for a simple two-dimensional system. For instance, suppose that d_{11} can not be properly compensated by c_{11}. The result is that the system can not be designed entirely within the non-interaction theory. However, a practical system can be found if the designer will accept some interaction. Furthermore, some of the non-interaction techniques can be used once the difficulty has been removed.

In finding a practical system it is probably best to start with the basic expression for d_{11}. This is:

$$d_{11} = a_{11} c_{11} + a_{12} c_{21} \tag{4-14}$$

In as much as it is impossible or impractical to compensate d_{11} with c_{11}, then c_{11} may in general be chosen arbitrarily. However, in some instances c_{21} may be chosen completely arbitrary and hence the arbitrariness of c_{11} will be restrained to a certain type of transfer function which results in $1 + d'_{11}$ having no zeroes in the right half p plane. In addition, certain simplifying assumptions in $\frac{1}{a^{-1}_{11}}$ may result in restrictions on c_{11}. In any event, c_{11} assumes a new description as c'_{11}. Since it may be necessary to change c_{21} from the transfer function dictated by non-interaction theory, it now becomes c'_{21}. Therefore,

$$1 + d'_{11} = 1 + a_{11} C'_{11} + a_{12} C'_{21} \tag{4-15}$$

The procedure is to make $1 + d'_{11}$ so that there are no zeroes in the right half p plane. Note that $1 + d_{11}$ will have zeroes in the right half p plane because $d_{11} = C_{11}/a^{-1}_{11}$ can not be properly compensated so that $Z = 0$.

In as much as the a_{ij} are fixed by the physical system it is much easier to design c'_{11} and/or c'_{21} so that there are no right half p plane zeroes in the resulting $1 + d'_{11}$. As a first approximation, write c'_{21} in the form dictated by non-interaction theory, which is:

$$C_{21} = -\frac{a_{21}}{a_{22}} C'_{11} \tag{4-16}$$

In as much as c'_{11} is arbitrary in some sense, make it as simple as desired. Then substitute into equation (4-15) and apply Routh's criteria to the resulting polynomial to determine whether or not these are zeroes in the right half p plane. If this test fails, alter c'_{21} by a factor, a, where a is a polynomial in the operator p or a ratio of polynomials. Again, test the resulting polynomial for positive

roots. Eventually, a q will be chosen such that $1 + d'_{11}$ has no zeroes in the right half p plane. In addition, if q is chosen so that the "break" frequencies are above the system natural frequencies then the interaction will be restricted to these higher frequencies.

Having removed the difficulty in d_{11}, attention can be focused on d_{22}. An attempt should be made to design it in accordance with the general theory of non-interacting systems. If no compensation difficulties are experienced then c_{22} and c_{12} will be dictated by the general theory. In addition, $1 + d_{22} = 1 + d'_{22}$ and there will be no right half p plane zeroes. If compensation difficulties are encountered then follow the procedure outlined for d'_{11}.

Naturally, the system has some interaction which appears through the d'_{12} and d'_{21} transfer functions. In fact, for the case presented here

$$d'_{12} = a_{11} C'_{12} + a_{12} C'_{22} = 0 \quad \left\{ \text{since } \begin{matrix} C'_{12} = C_{12} \\ C'_{22} = C_{22} \end{matrix} \right\} \tag{4-17}$$

$$d'_{21} = a_{21} C'_{11} + a_{22} C'_{21} \neq 0 \tag{4-18}$$

Due to interaction, the overall system closed loop stability is dictated by the zeroes of:

$$(1 + d'_{11})(1 + d'_{22}) - d'_{12} d'_{21} = 0 \tag{4-19}$$

Determining the stability through this equation has already been discussed at length in the previous section. In the event that there were zeroes of equation (4-19) in the right half p plane they would stem from the interaction terms, d'_{12} and d'_{21}. Thus, it would be necessary to choose another q factor so that

both the $1 + d'_{ii}$ and $1 - \frac{d'_{12} d'_{21}}{(1+d'_{11})(1+d'_{22})}$ contributed no right half p plane zeroes to equation (4-19).

In the case considered here $d'_{12} = 0$. Thus, system stability is assured if $(1 + d'_{11})(1 + d'_{22})$ have no zeroes in the right half p plane. Actually, $d'_{12} \cong 0$ due to the slight deviations in a_{ij} and c_{ij} of equation (4-17). However, it is not expected that such slight interaction through this term would cause instability when coupled with d'_{21}.

In summary, when compensation difficulties arise and interchanging the rows of the δ vector does not alleviate the problem, then it is necessary to deviate from the non-interaction theory to the extent that there are no zeroes in the right half p plane of the resulting characteristic equation of the system, namely:

$$(1 + d'_{11})(1 + d'_{22}) - d'_{12} d'_{21} = 0$$

Naturally, interaction will be introduced through the d'_{ij} transfer functions. In the case considered here, the amount of interaction as a function of frequency can be found from the following transfer functions:

$$y_1/x_2 = \frac{d'_{12}}{|D'|} \tag{4-20}$$

$$y_2/x_1 = \frac{d'_{21}}{|D'|} \tag{4-21}$$

where $|D'| = (1 + d'_{11})(1 + d'_{22}) - d'_{12} d'_{21}$

Chapter V

The Synthesis of an Aircraft Control System

In present day aircraft, the auto pilot performs many functions in the integrated control system. Although the system inputs are generally restricted to combinations of elevator, rudder, aileron and throttle, the system controlled outputs are many.* For instance, an auto pilot system could control the aircraft velocity vector through the control of the pitch, yaw and roll rates. Other systems could control either altitude, angle of attack, airspeed, glide path, Mach number, lateral acceleration, turning rate, etc., or combinations of the foregoing. It is the intent here only to point out that the auto pilot can be used to control many variables and each subsystem of the auto pilot can control as many outputs as it has inputs.

In order to point up the theoretical analysis and the practical problems associated with a design based on the general theory of non-interacting controls, an auto pilot subsystem shall be synthesized in this section. Naturally, the goal of such a study is to arrive at a controller design that permits good system performance as well as the desired non-interaction. However, as was pointed out in the preceding sections, a perfect non-interaction over all frequencies is not obtainable nor is it necessarily required. In addition, complex physical systems such as an aircraft could easily require impractical or impossible controller compensation when applying the general non-interaction theory. Thus, the designer may have to accept some interaction in order to preserve system stability. Furthermore, he may be forced, at least in part, to deviate from the general theory in order to design a practically realizable system with

desirable characteristics.

The subsystem to be synthesized is one dealing with longitudinal control in which the inputs are elevator and throttle deflections. The controlled outputs are glide path angle, $\overline{\theta_W}$, and indicated airspeed, $\overline{\mathcal{V}_P}$. It should be emphasized that this study was chosen only as a means of lending practical significance to the theory and is not considered as a demonstration of a specific auto pilot design.

The assumed aircraft and flight condition is an F-86 "Saberjet" making an approach down a 3^{o} glide slope. Small perturbations from symmetric equilibrium flight will be assumed thus permitting the use of the linearized longitudinal equations in the ensuing analysis. Upon determination of the controller design for the best system performance, the synthesized system will be studied on the differential analyzer. Such a study should clearly demonstrates the applicability of the theory, error analysis and the practical system problems.

The large motion longitudinal equations of motion for an aircraft using a combined body-wind axis system are shown in Ref. 5 to be:

$$\sum F_{wx} = 0 = -m\dot{V}_P + F_{wx} = -m\dot{V}_P + P_x\cos\alpha + P_z\sin\alpha + X_s - mg\sin\theta_w \tag{5-1}$$

$$\sum F_{wz} = 0 = mV_P Q_w + F_{wz} = mV_P Q_w - P_x\sin\alpha + P_z\cos\alpha + Z_s + mg\cos\theta_w \tag{5-2}$$

$$\sum M_y = 0 = -I_{yy}\dot{Q} + [I_{zz} - I_{xx}]RP + I_{zx}[R^2 - P^2] + M + \Upsilon_y + I_{xy}(\dot{P} + QR) + I_{yz}(\dot{R} - PQ) \tag{5-3}$$

The linearization of these equations is carried out in Appendix I. In addition, a fourth linearized equation is obtained so that there are four simultaneous equations involving the four output variables, $\overline{v_p}$, $\overline{\alpha}$, q, $\overline{\theta_w}$ and the inputs, $\overline{\delta_T}$ and $\overline{\delta_e}$.

It is desired to determine the transfer functions that relate $\overline{v_p}$ and $\overline{\theta_w}$ to $\overline{\delta_T}$ and $\overline{\delta_e}$. It should be clear from the theory that these transfer functions are the $\overline{a}_{ij}$ elements of the $\overline{A}$ (aircraft) matrix. The mathematical steps taken to determine these elements from the four simultaneous equations are completely outlined in Appendix I.

The elements, $\overline{a}_{ij}$, are merely a ratio of polynomials in the operator p or D. However, they derive their existence from the coefficients of the linearized longitudinal equations. These coefficients are composed of thrust terms, gravity terms and aerodynamic force and moment terms expressed through the aircraft stability derivatives and trim lift and drag coefficients.

The stability derivatives for the F-86 were obtained from Ref. 6 for flight at sea level at 163 knots. They are listed in Appendix II along with other physical data pertaining to the F-86. Calculation of the trim lift and drag coefficients as well as the b_{ij} and f_{ij} coefficients of equations (I-9) through (I-12) are also found in Appendix II. Knowing the b_{ij} and f_{ij} coefficients permitted the calculation of the $\overline{a}_{ij}$ elements from equations (I-18) through (I-21) of Appendix I. The results are:

$$\overline{a}_{11} = 10.22 \left[\frac{(D-.012)(D^2+1.422D+3.4771)}{(D^2+1.4144D+3.4534)(D^2+.0156D+.0245)} \right]$$

$$\bar{a}_{12} = -.885 \left[\frac{(D-150)(D+1.1)}{(D^2+1.4144D+3.4534)(D^2+.0156D+.0245)} \right]$$

$$\bar{a}_{21} = .00625 \left[\frac{(D+1.409)(D^2+.701D+3.087)}{(D^2+1.4144D+3.4534)(D^2+.0156D+.0245)} \right]$$

$$\bar{a}_{22} = .07 \left[\frac{(D-.00717)(D+7.65)(D-8.5)}{(D^2+1.4144D+3.4534)(D^2+.0156D+.0245)} \right]$$

The two quadratic factors in the denominator of the $\bar{a}_{ij}$ terms provide information as to the period and damping of the short period and phugoid motions. The period and damping factors obtained are:

Short period motion	Phugoid motion
$T_s = \frac{2\pi}{\omega_{n_s}} = 3.38$ sec.	$T_p = \frac{2\pi}{\omega_{n_p}} = 40$ sec.
$\zeta_s = .382$	$\zeta_p = .0842$

It may be seen from Figs. 1 and 5 that in vector-matrix notations:

$$y = T\bar{A}S\,\delta \qquad \text{where} \qquad T = (t_{ii}) \quad S = (s_{ii}) \quad \bar{A} = (\bar{a}_{ij})$$

For the specific problem under study:

$$y_1 = \nu_p \;;\; \bar{y}_1 = \bar{\nu}_p \;;\; \bar{\delta}_1 = \bar{\delta}_T \;;\; \delta_1 = \delta_T$$

$$y_2 = \theta_w \;;\; \bar{y}_2 = \bar{\theta}_w \;;\; \bar{\delta}_2 = \bar{\delta}_e \;;\; \delta_2 = \delta_e$$

Therefore:

$$\nu_p = (t_{11}\,\bar{a}_{11}\,s_{11})\,\delta_T + (t_{11}\,\bar{a}_{12}\,s_{22})\,\delta_e = a_{11}\,\delta_T + a_{12}\,\delta_e \qquad (5\text{-}4)$$

$$\theta_w = (t_{22}\,\bar{a}_{21}\,s_{11})\,\delta_T + (t_{22}\,\bar{a}_{22}\,s_{22})\,\delta_e = a_{21}\,\delta_T + a_{22}\,\delta_e \qquad (5\text{-}5)$$

To determine the a_{ij} terms of the above equations, it is necessary to specify

the elements of the diagonal servo and transducer matrices, S and T respectively. This has been done in Appendix III. In addition, reasons governing the specification of these elements or transfer functions are included in the appendix. With this knowledge, the a_{ij} terms of equations (5-4) and (5-5) are found to be:

$$a_{11} = 10.22\left[\frac{(D-.012)(D^2+1.422D+3.4771)}{(D^2+1.4144D+3.4534)(D^2+.0156D+.0245)(2D+1)(.2D+1)}\right]$$

$$a_{12} = -.885\left[\frac{(D-150)(D+1.1)}{(D^2+1.4144D+3.4534)(D^2+.0156D+.0245)(.04D+1)(.2D+1)}\right]$$

$$a_{21} = .00625\left[\frac{(D+1.409)(D^2+.701D+3.087)}{(D^2+1.4144D+3.4534)(D^2+.0156D+.0245)(2D+1)}\right]$$

$$a_{22} = .07\left[\frac{(D-.00717)(D+7.65)(D-8.5)}{(D^2+1.4144D+3.4534)(D^2+.0156D+.0245)(.04D+1)}\right]$$

It has been pointed out that $d_{ii} = c_{ii}/a_{ii}^{-1}$ is an open loop transfer function for one "channel" of the auto pilot system. From Fig. 5, it may be seen that the open loop vector-matrix equation is $y = AC\epsilon$. It follows that:

$$d_{11} = Y_{0_1} = y_1/\epsilon_1 = c_{11}/a_{11}^{-1} = \frac{c_{11}|A|}{a_{22}} \equiv \frac{N_P}{\epsilon_{N_P}} \tag{5-6}$$

$$d_{22} = Y_{0_2} = y_2/\epsilon_2 = c_{22}/a_{22}^{-1} = \frac{c_{22}|A|}{a_{11}} \equiv \frac{\theta_w}{\epsilon_{\theta w}} \tag{5-7}$$

It is evident that equations (5-6) and (5-7) are the open loop transfer functions for the velocity and pitch angle "channels". It is desired that these channels be closed loop stable as well as having desirable response characteristics. An attempt is made to design the controller transfer functions c_{11} and c_{22} so that these requirements are met. If they are satisfied, then the other controller transfer functions, c_{12} and c_{21} are determined from equation (2-14).

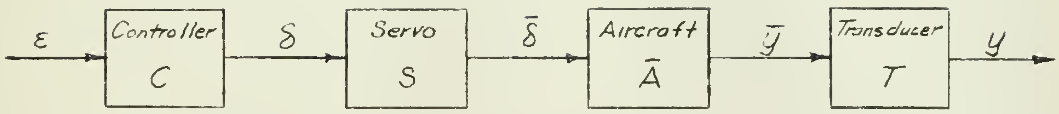


Fig. 5 The Open Loop Control System Configuration

The determinant of the A matrix was found to be:

$$|A| = |a_{11}a_{22} - a_{12}a_{21}|$$

$$|A| = .71\left[\frac{(D+7.74)(D-8.587)(D^2+1.415D+3.456)(D^2+.0187D+.0214)}{(2D+1)(.2D+1)(.04D+1)(D^2+1.4144D+3.4534)^2(D^2+.0156D+.0245)}\right]$$

Substituting into equations (5-6) and (5-7) and cancelling equal or essentially equal terms in numerator and denominator (excepting terms in right half plane) provided:

$$Y_{0_1} = C_{11}(10.2)\left[\frac{(p-8.587)}{(p-.00717)(p-8.5)(2p+1)(.2p+1)}\right] \tag{5-8}$$

$$Y_{0_2} = C_{22}(1.333)\left[\frac{(.129p+1)(.1165p-1)}{(p-.012)(.288p^2+.409p+1)(.04p+1)}\right] \tag{5-9}$$

In deciding what c_{11} should be in order that $\frac{d_{11}}{1+d_{11}}$ be closed loop stable with good response characteristics, Nyquist plots had to be used. A sketch of the Nyquist plot of $\frac{Y_{0_1}}{C_{11}K}$ is shown in Fig. 6. There are two poles in the right half p plane of $\frac{Y_{0_1}}{C_{11}K}$. Nyquist stability criteria indicates that $N = +2$ for $Z = 0$. This implies that c_{11} must be chosen such that the Nyquist plot of the resulting Y_{0_1} or d_{11} must encircle the critical point, -1, 0, twice in a counter clockwise direction. Without doubt, the required compensation presents a rather formidable problem. In fact no practical compensation could be found. Thus, it is clear that the system will be closed loop unstable in that it can not be properly compensated by c_{11}.

In an attempt to alleviate the above design problem, the rows of the δ vector were interchanged. As a result, it was possible to compensate d_{11} in accordance with the general non-interaction theory. However, compensation

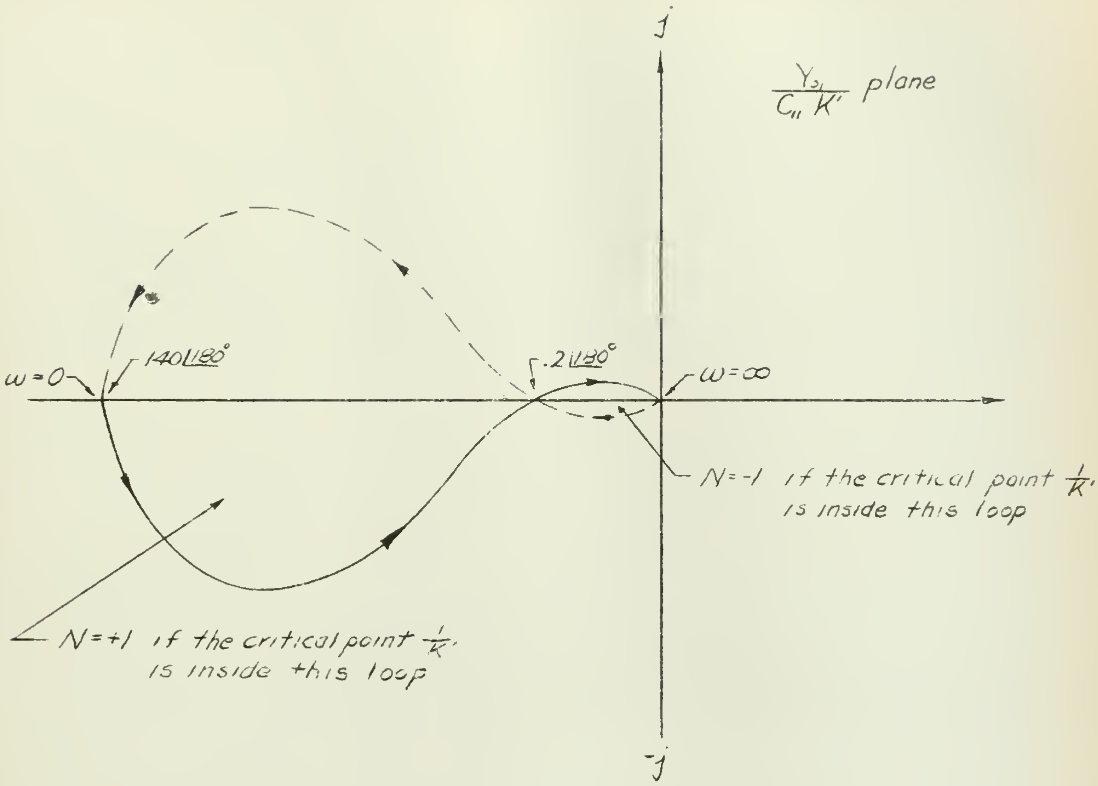


Fig. 6 A Sketch of the Nyquist Plot for the Transfer Function, $Y_{01}/C_{11}K$

problems arose in Y_{o_2} or d_{22} and no practical means of compensating were found. Thus, this method failed.

The next approach was to alter d_{11} with a factor q such that there were no zeroes of $1 + d'_{11}$ in the right half p plane. This approach is discussed at length in the section on system problems. From that section, it is known that a more general form for $1 + Y_{o_1}$ or $1 + d'_{11}$ is given by equation (4-15) which is repeated here:

$$1+d'_{11} = a_{11} C'_{11} + a_{12} C'_{21} + 1 \tag{4-15}$$

It was pointed out in Chapter IV that c'_{11} or c'_{21} could be somewhat arbitrary when d_{11} could not be properly compensated by the non-interaction design methods. In the case at hand, instead of making c'_{11} completely arbitrary, let us assume that the p-8.5 pole is cancelled by the p-8.587 zero in equation (5-8). Of course, it is understood that this is incorrect in that one of the right half p plane poles are being neglected. Nevertheless, compensating the remaining Y_{o_1} with P = 1 results in c'_{11} being:

$$C'_{11} = .5\left(\frac{2p+1}{.1p+1}\right) \tag{5-10}$$

Naturally, this is not the necessary compensation for closed loop stability of $\frac{d_{11}}{1+d_{11}}$ as dictated by non-interaction theory.

As a first approximation to c'_{21}, let it be dictated by the equation:

$$C'_{21} = -\frac{a_{21}}{a_{22}} C'_{11}$$

Substituting for a_{21}, a_{22} and c'_{11} results in:

$$C_{21}' = \frac{-.00292(.711p+1)(.04p+1)(.324p^2+.227p+1)}{(.1p+1)(p-.00717)(.1309p+1)(.1178p-1)} \qquad (5-11)$$

Substituting equations (5-10) and (5-11) as well as a_{11} and a_{22} into equation (4-15) and applying Routh's stability criteria to the resulting ninth order polynomial indicates that there are zeroes in the right half p plane. However, by altering c_{21}' above by a factor q_2, where $q_2 = -(.1178p - 1)/(.04p + 1)$, the new c_{21}' becomes:

$$C_{21}' = \frac{.00292(.711p+1)(.324p^2+.227p+1)}{(p-.00717)(.1p+1)(.1309p+1)} \qquad (5-12)$$

Going through the same substitution procedure and applying Roth's criteria indicates that there are no zeroes of $1 + d_{11}'$ in the right half p plane. It follows that the velocity "channel," Y_1, is now closed loop stable. In addition, interaction is present in the system because the c_{11}' and c_{21}' of equations (5-10) and (5-12) are different from the c_{11} and c_{21} that would have been dictated by following the non-interaction theory.

The d_{22} transfer function was determined from the non-interaction theory. It is shown in equation (5-9). A Nyquist plot of $-Y_{o2}/C_{22}K$ is shown in Fig. 7. By making $C_{22} = .5\left[\frac{1.2p+1}{.02p+1}\right]$, d_{22}(or Y_{o2}) was properly compensated so that there were no zeroes in the right half p plane. Furthermore, the compensation was chosen so as to give good transient response.

Having determined c_{22} in accordance with the non-interaction theory, then it was permissible to find c_{12} from equation (2-14). Therefore:

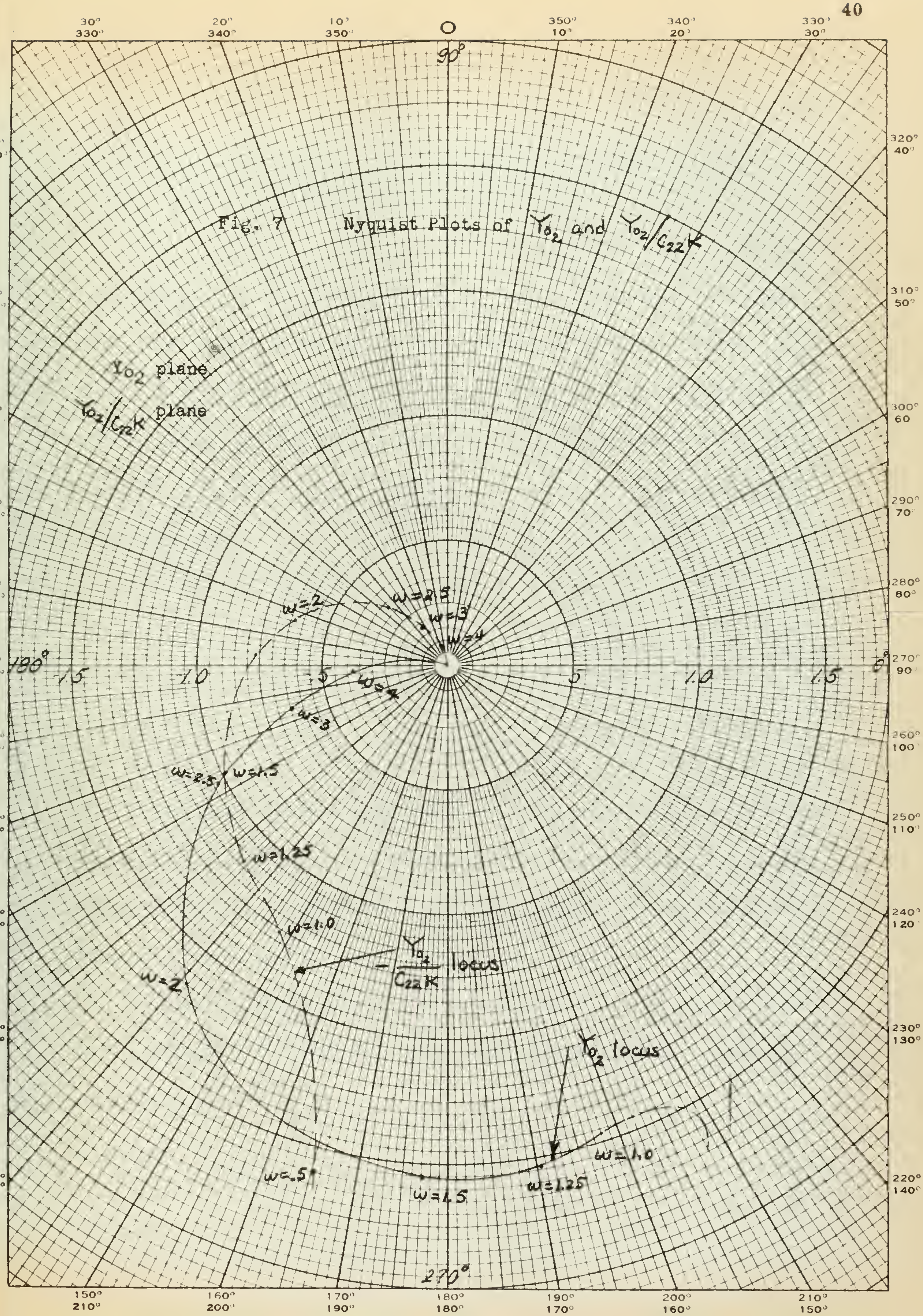


Fig. 7 Nyquist Plots of Y_{o_2} and $Y_{o_2}/C_{22}K$
Y_{o_2} plane
$Y_{o_2}/C_{22}K$ plane
$-\frac{Y_{o_2}}{C_{22}K}$ locus
Y_{o_2} locus
ω=2
ω=2.5
ω=3
ω=4
ω=1.5
ω=1.25
ω=1.0
ω=.5
1.5
1.0
.5

$$C_{12} = \frac{-a_{12}c_{22}}{a_{11}} = \frac{-2.045\,(1.2p+1)(.91p+1)(2p+1)(.0067p-1)}{(p-.012)(.04p+1)(.02p+1)(.288p^2+.409p+1)} \tag{5-13}$$

In as much as terms such as (.00667p − 1), (.02p + 1) and (.04p + 1) are only effective at the higher frequencies, c_{12} can be simplified by neglecting them. Naturally this will give rise to interaction at higher frequencies. But this is of no real consequence provided that these gross deviations do not impair the system stability through the $d'_{12}d'_{21}$ term of the system characteristic equation. In the problem considered here the stability is not impaired by the interaction terms.

In light of the foregoing:

$$C'_{12} = g_1 C_{12} = \frac{2.045\,(1.2p+1)(.91p+1)(2p+1)}{(p-.012)(.288p^2+.409p+1)} \tag{5-14}$$

where

$$g_1 = \frac{(.04p+1)(.02p+1)}{-(.0067p-1)}$$

Attenuation plots of c'_{11}, c_{22} and the c_{ij} of equations (5-10) through (5-14) are shown in Fig. 8. One may readily see that the effect of neglecting the high frequency terms is to level off the gain at high frequencies, such frequencies being above the natural frequency of the overall system.

The amount of interaction contributed through d'_{12} and d'_{21} can be found from equations (4-20) and (4-21). In the case considered here, d'_{12} and d'_{21} are as follows:

$$d'_{12} = a_{11}\, g_1\, C_{12} + a_{12}\, C_{22} \tag{5-15}$$

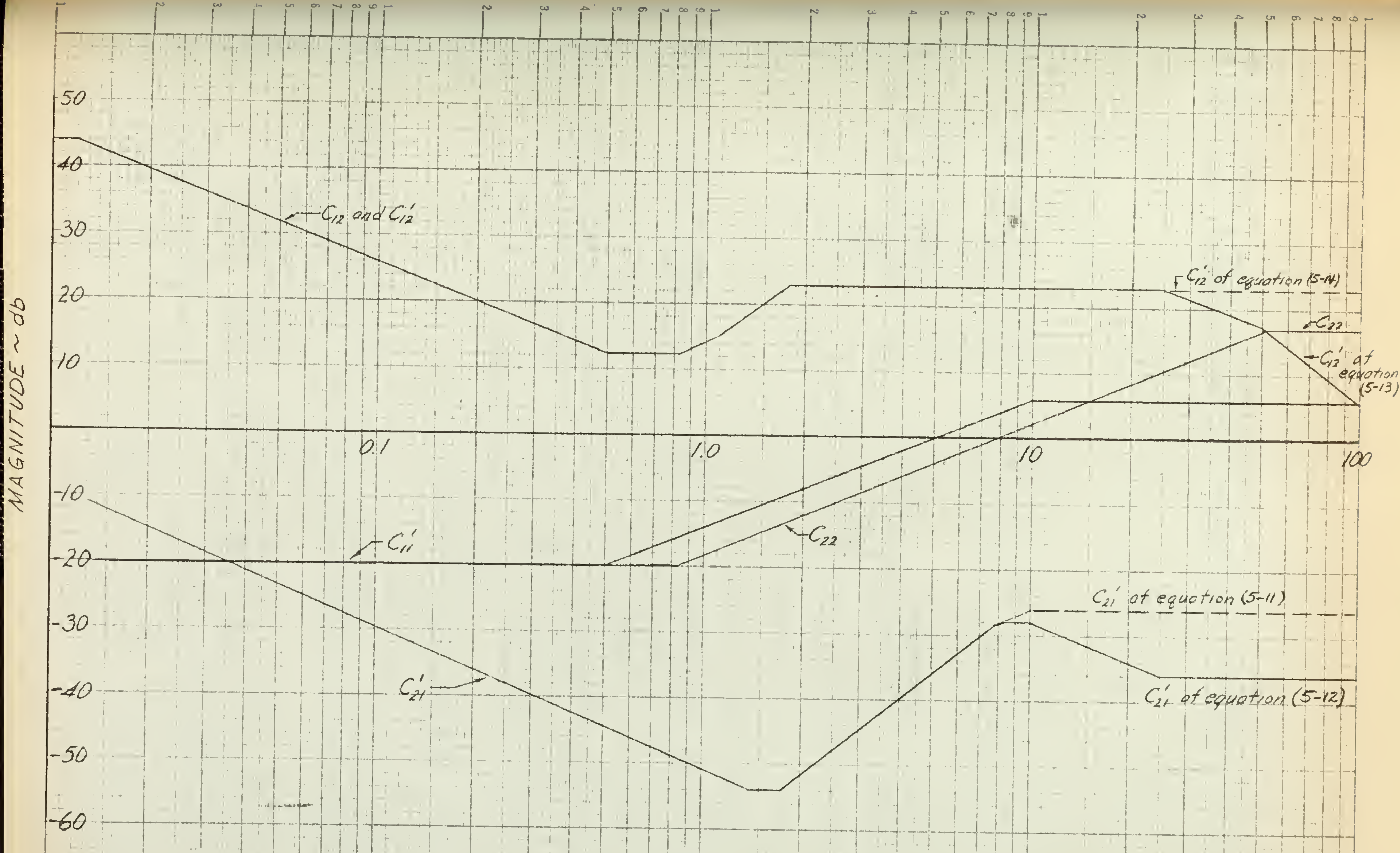


Fig. 8 Magnitude-Frequency Characteristics of the Controller

$$d'_{21} = a_{21} c'_{11} + c'_{21} a_{22} \tag{5-16}$$

where c'_{21} is given by equation (5-12) and $c_{12}q_1$ is given by equation (5-14).

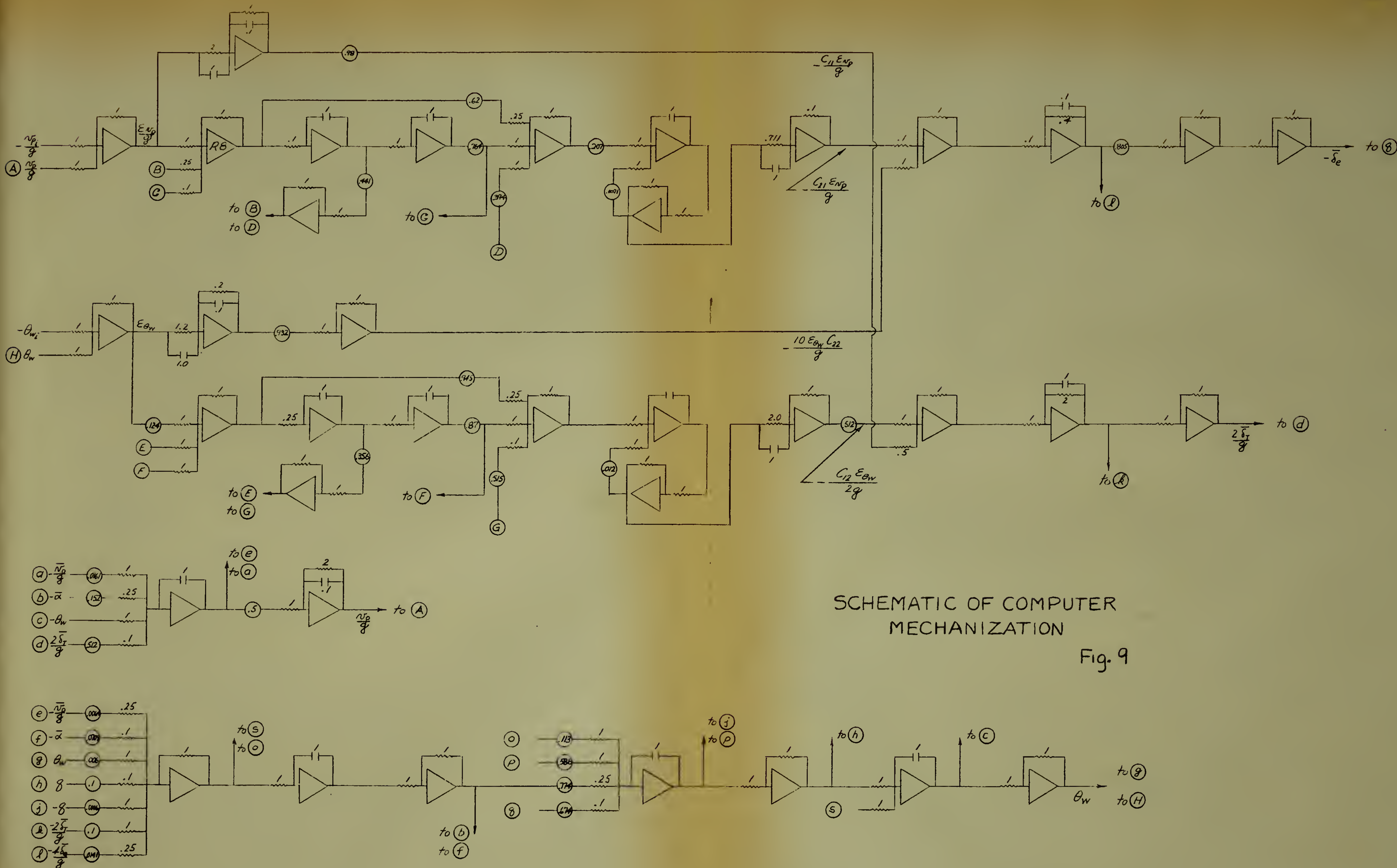
Computer Simulation

With the c_{ij} designed, the auto pilot system was mechanized for study on the Reeves Electronic Analog Computer (REAC). The mechanization used is shown in Fig. 9.

System response to step inputs of command velocity, $\frac{v_{p_i}}{g}$, and command glide path angle, θ_{w_i}, are shown in Figs. 10 and 11. Note that the interaction is essentially zero except at high frequency. However, this was expected in that high frequency terms in c'_{12} and c'_{21} were neglected. There is, although not evident in Figs. 10 and 11, a low amplitude damped oscillation that affects the steady state character of $\frac{v_p}{g}$ and θ_w. These oscillations are the direct result of disturbance inputs, u_α, that enter the system through the c_{ij}. In fact, Figs. 12(a) and 12(b) show these oscillations in $\frac{v_p}{g}$ and θ_w when there are no inputs other than disturbances in the system.

As a means of verifying the magnitude, frequency and damping of the low amplitude oscillation, a step input, x_2, was introduced into c_{21} through the input to amplifier R-8 of Fig. 9. The effect of the disturbance input on $\frac{v_p}{g}$ and θ_w is shown in Figs. 13(a) and 13(b).

An analytical study of the system was made in order to determine the relation between $\frac{v_p}{g}$ and x_2 and also to verify the traces seen in Fig. 13 as those that stem from disturbance inputs. In the case where there are no desired



SCHEMATIC OF COMPUTER MECHANIZATION

Fig. 9

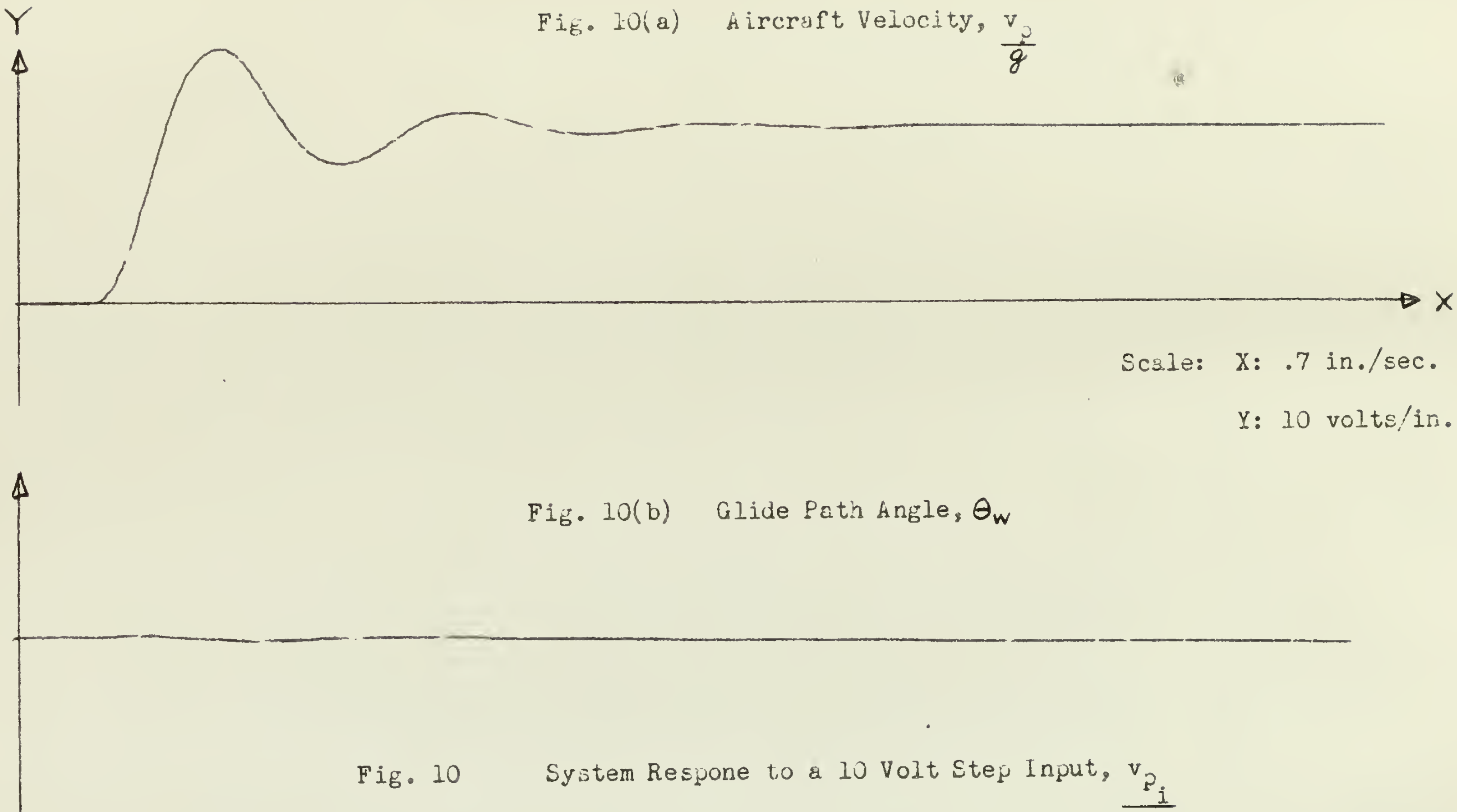


Fig. 10(a) Aircraft Velocity, $\frac{v_p}{g}$

Scale: X: .7 in./sec.

Y: 10 volts/in.

Fig. 10(b) Glide Path Angle, θ_w

Fig. 10 System Respone to a 10 Volt Step Input, $\frac{v_{p_i}}{g}$

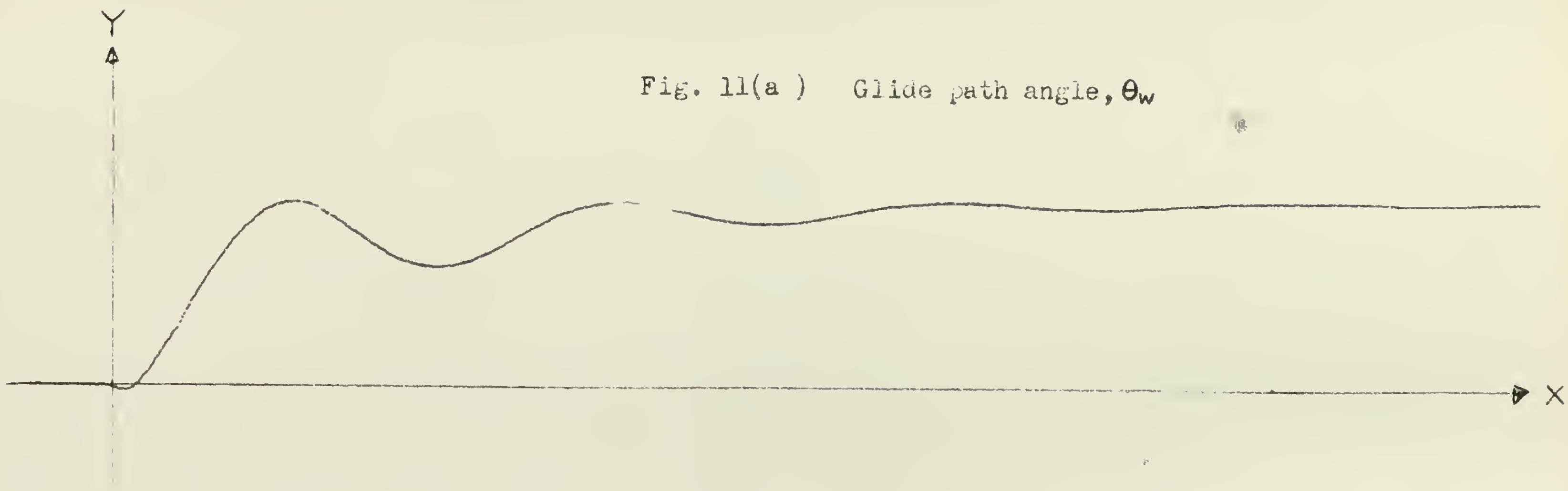


Fig. 11(a) Glide path angle, θ_w

Scale: X: 0.5 in./sec

Y: 10 volts/in

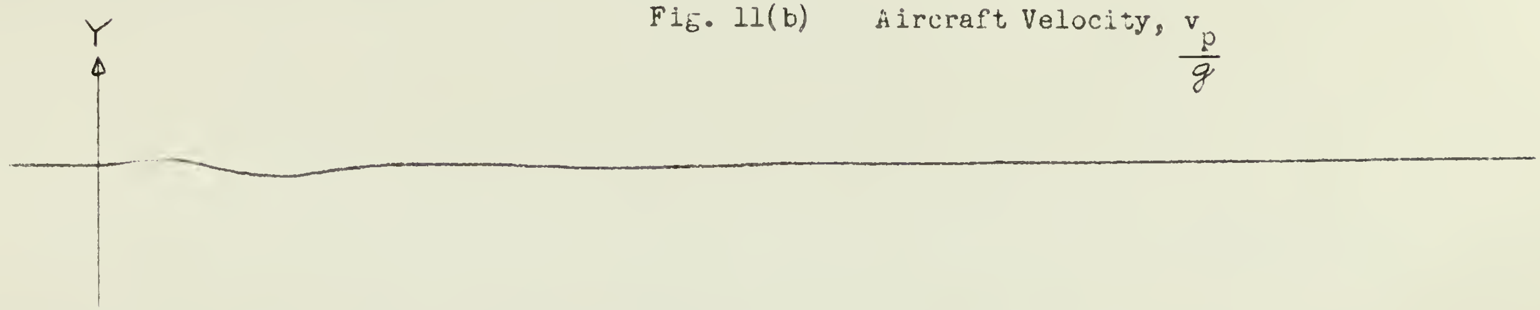


Fig. 11(b) Aircraft Velocity, $\frac{v_p}{g}$

Fig. 11 System Response to a 10 Volt Step Input, θ_{w_i}

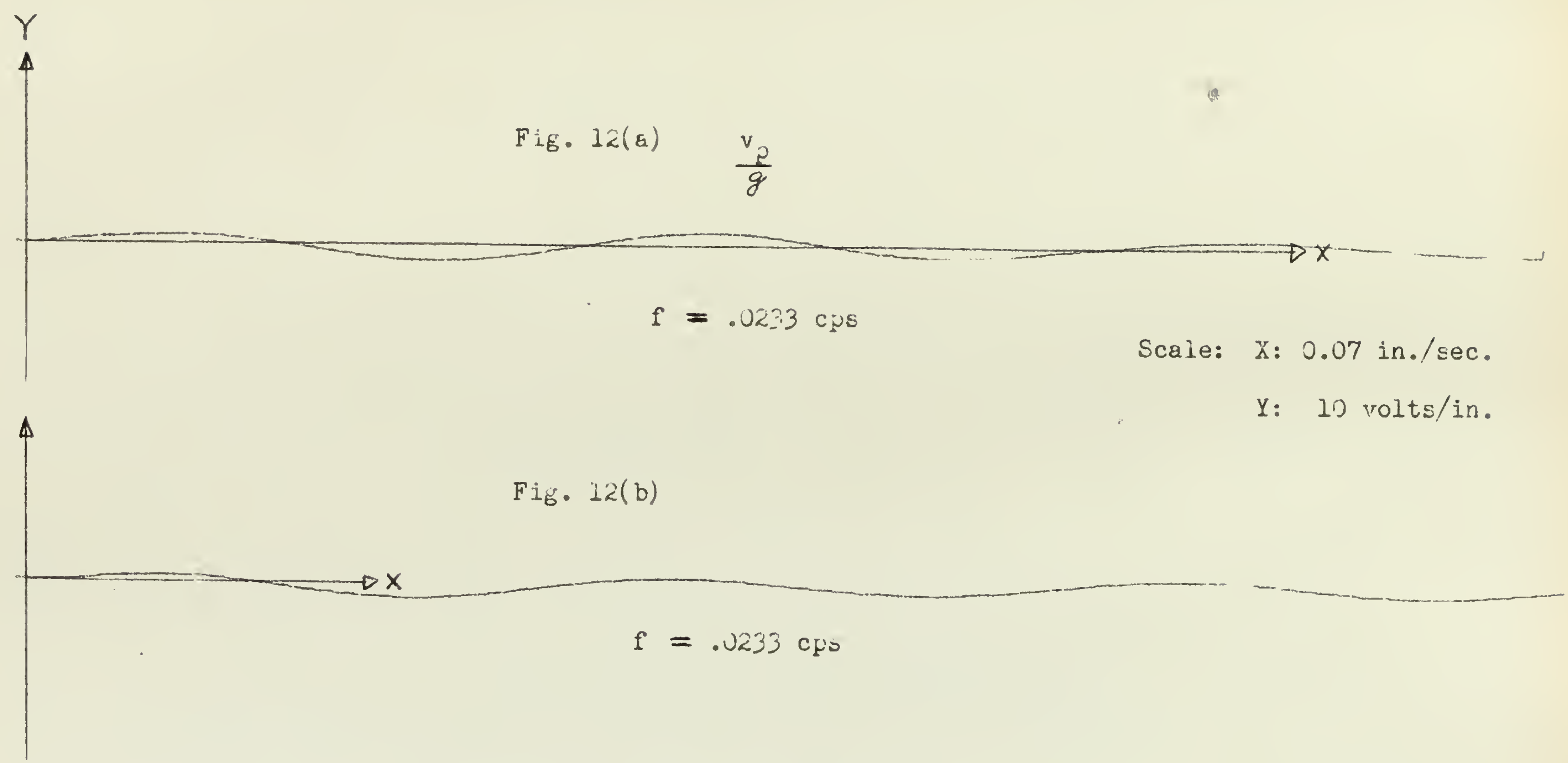
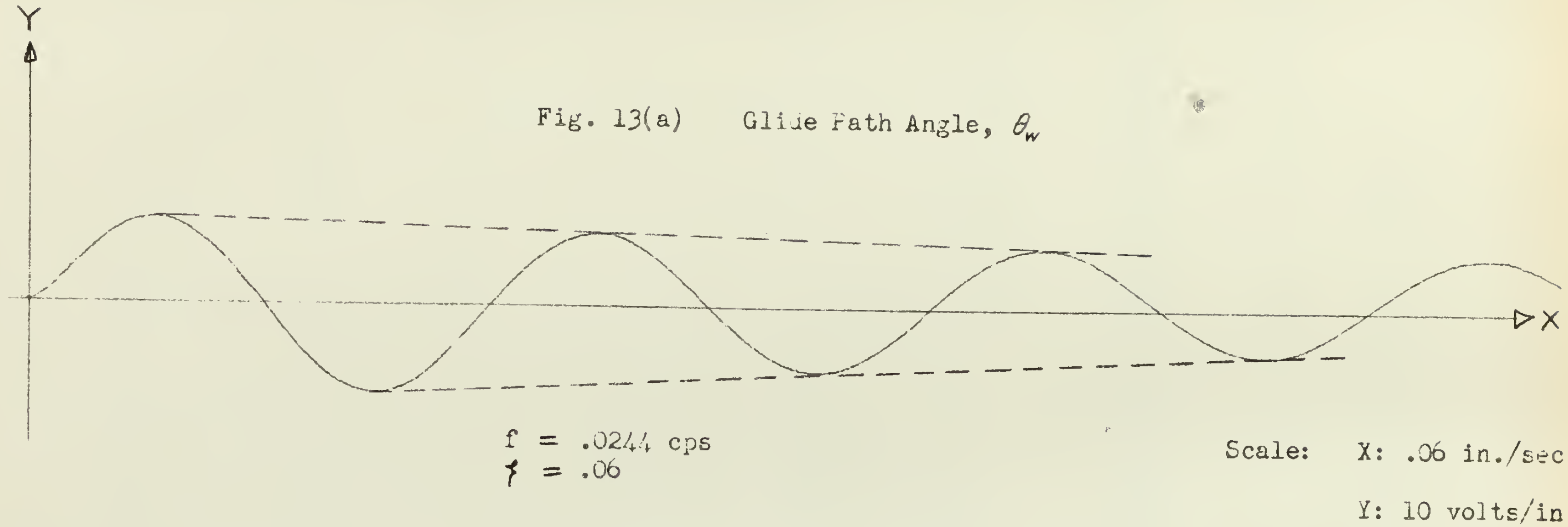


Fig. 12 System Response Due to Actual Disturbance Inputs in the Control System; $v_{p_i} = \theta_{w_i} = 0$



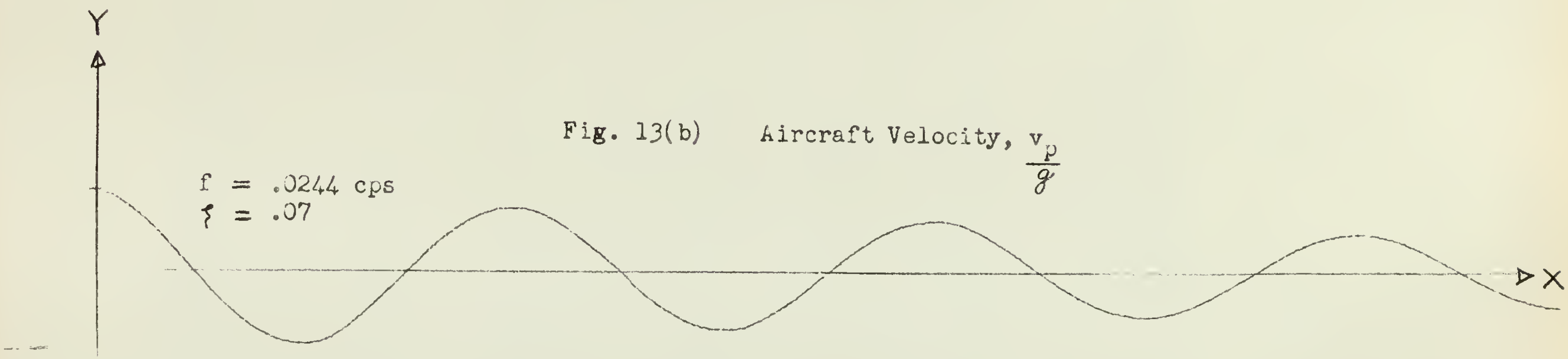


Fig. 13 System Response to a 5 Volt Step Disturbance Input to c_{21}

$(v_{p_i} = \theta_{w_i} = 0)$

perturbations from steady state condition, then $\mathcal{N}p_i = 0$. Therefore, it may be shown that:

$$\mathcal{N}p = \frac{a_{12} C_{21}}{1 + a_{11} C_{11} + a_{12} C_{21}} X_2 \tag{5-17}$$

Substituting into equation (5-17) and simplifying results in:

$$\mathcal{N}p = \frac{(24.7)(.711p+1)(p+1.1)(p-150)(.324p^2+.227p+1)}{\left[p^9 + 49.1p^8 + 800p^7 + 5970p^6 + 27{,}050p^5 + 90{,}400p^4 + 131600p^3 + 171{,}000p^2 + 5150p + 4100\right]} \tag{5-18}$$

where the static gain is unity.

Application of Routh's criteria to the ninth order polynomial in the denominator of equation (5-18) indicates that there are no positive roots. Therefore, the system is stable for the disturbance input, x_2. In addition, it may be shown that $(p^2 + .0194p + .0210)$ is a factor of the ninth order polynomial. Furthermore, it is the only factor which has a low characteristic frequency. In fact this frequency is .0231 cycles per second and the damping factor is $\zeta = .067$. It is quite apparent that this quadratic factor alone accounts for the lightly damped high period oscillation seen in Fig. 13. Naturally, similar analysis can be made for other disturbance inputs into the system.

The low amplitude oscillation discussed above as well as other responses to disturbance inputs can have a detrimental effect on the system due primarily to their magnitude and damping. The addition of a high gain loop would introduce gain prior to the introduction of noise and hence make the system less sensitive to the disturbance inputs.

Chapter VI

Conclusions

1. In theory, control systems based on non-interaction of control variables are feasible but somewhat complex. For complete non-interaction, the theory requires precise mathematical knowledge of the behavior of the system to be controlled and exact mechanization of the controller transfer functions. In reality, these requirements generally can not be met but can be approached to the point that the resulting interaction is negligible for all frequencies.

2. Small deviations, e_i , in a_{ij} due to insufficient knowledge of the physical system and/or in c_{ij} due to the inability to mechanize transfer functions exactly should not impair the stability of the system if the d_{ii}, or open loop transfer functions, are designed with a reasonable amount of stability.

3. Gross deviations in c_{ij}, so as to obtain a simpler mechanization can result in considerable interaction and possible instability in the closed loop system. If deviations are restricted to terms effective only at frequencies well above the natural frequency of the system, then the interaction will be restricted to these frequencies. However, system stability can be impaired through the interaction terms now present in the characteristic equation of the overall system. This equation should be investigated for zeroes in the right half complex plane.

4. System errors due to disturbances entering the system can be large. However, these large errors can be reduced considerably by addition of a suitable high gain loop.

5. The system error equation indicates that the overall system can be

stable for command inputs, x_i, and yet be unstable for the disturbance inputs, $\mathcal{U}_\alpha$ and $\mathcal{N}_\beta$. In any event, a complete error analysis for disturbance inputs should be made.

6. Zeroes in the right half p plane of the co-factors of a_{ii}^{-1} can create compensation problems through the open loop relation, $d_{ii} = \frac{c_{ii}}{a_{ii}^{-1}}$. Naturally, c_{ii} must be chosen so that there are no zeroes in the right half p plane of $1 + d_{ii}$. Making the proper choice of c_{ii} can be a formidable problem in some instances with the result that no practical compensation can be found.

7. Interchanging the rows of the control vector, δ, will alleviate the impractical compensation problems in some cases.

8. When compensation problems are encountered and changing the rows of the δ vector does not alleviate the problem, the non-interaction theory can not be used for the design of that particular "channel" or open loop part of the system which is causing the problem. Some other approach must be used to make the "channel" closed loop stable. Normally, the end result of deviation from the non-interaction theory is system interaction, which, it is hoped, can be made small.

APPENDIX I

LINEARIZED LONGITUDINAL EQUATIONS AND DETERMINATION OF THE ELEMENTS OF THE A MATRIX

A. Linearized Longitudinal Equations

The equations for longitudinal motion in the general or large motion case in a combined body-wind axis system have been shown in Ref. 5 to be:

(I-1) $$\sum F_{wx} = 0 = -m\dot{V}_p + F_{wx} = -m\dot{V}_p + P_x \cos\alpha + P_z \sin\alpha + X_s - mg\sin\Theta_w$$

(I-2) $$\sum F_{wz} = 0 = +mV_pQ_w + F_{wz} = mV_pQ_w - P_x \sin\alpha + P_z \cos\alpha + Z_s + mg\cos\Theta_w \ .$$

(I-3) $$\sum M_Y = 0 = -I_{yy}\dot{Q} + [I_{zz} - I_{xx}]RP + I_{zx}[R^2 - P^2] + M + \mathcal{T}_y + I_{yz}[\dot{R} - PQ] + I_{xy}[\dot{P} + QR]$$

Considering small perturbations from a steady state longitudinal flight condition, the above equations can be linearized as follows:

1. Assume the following conditions:

$$P = R = \beta = \Phi = 0 \quad ; \quad I_{yz} = I_{xy} = 0$$
$$Q_1 = Q_{w_1} = \dot{\alpha}_1 = 0 \quad ; \quad \mathcal{T}_y = 0$$

2. Assume the following steady state plus perturbation relations for the system variables:

$$V_p = V_{p_1} + \bar{v}_p \quad ; \quad Q_w = Q_{w_1} + q_w$$
$$\alpha = \alpha_1 + \bar{\alpha} \quad ; \quad \delta_e = \delta_{e_1} + \bar{\delta}_e$$
$$\Theta_w = \Theta_{w_1} + \bar{\theta}_w \quad ; \quad \delta_T = \delta_{T_1} + \bar{\delta}_T$$
$$Q = Q_1 + q$$

where $\bar{v}_P, \alpha, \bar{\theta}_w, q, q_w, \bar{\delta}_e, \bar{\delta}_T$ are the perturbation variables.

3. The dimensionless coefficients, X_S, Z_S, Y_S, N_S, L_S, and M_S are functions of $\alpha, \dot{\alpha}, \beta, P_S, Q_S$ and R_S. By using non-dimensional angular rates, $\bar{\dot{\alpha}}, \bar{P}, \bar{Q}$ and $\bar{R}$, then X_S, Z_S and M_S become functions of $\alpha, \bar{\dot{\alpha}}, \bar{Q}$ and control inputs only. Thus, neglecting the lesser important derivitives, the dimensionless coefficients become:

$$(I\text{-}4) \quad X_s = \tfrac{1}{2}\rho V_P^2 S\left[C_{X_\alpha}\alpha + C_{X_0}\right]$$

$$(I\text{-}5) \quad Z_s = \tfrac{1}{2}\rho V_P^2 S\left[C_{Z_\alpha}\alpha + C_{Z_0} + C_{Z_{\bar{Q}}}\bar{Q} + C_{Z_{\delta_e}}\bar{\delta}_e\right]$$

$$(I\text{-}6) \quad M_s = \tfrac{1}{2}\rho V_P^2 S\left[C_{M_\alpha}\alpha + C_{M_{\bar{\dot{\alpha}}}}\bar{\dot{\alpha}} + C_{M_0} + C_{M_{\bar{Q}}}\bar{Q} + C_{M_{\delta_e}}\bar{\delta}_e\right]$$

4. In the wind axis system, it is true that:

$$(I\text{-}7) \quad \dot{\theta}_W = Q_W$$

$$(I\text{-}8) \quad \dot{\alpha} = Q - Q_W$$

5. Linearizing equations (I-1) through (I-3) about the steady state condition (subscript one) and rearranging in operator form with $D = d/dt$ provides:

$$(I\text{-}9) \quad \underline{b_{11}\bar{v}_P + b_{12}\bar{\alpha} + b_{13}\bar{\theta}_W + b_{14}q = f_{11}\bar{\delta}_T + f_{12}\bar{\delta}_e}$$

where:

$$b_{11} = \frac{D}{g} - K_{vv} = \left[\frac{D}{g} - \frac{\rho V_{P_1} S}{mg}\left\{C_{X_\alpha}\alpha_1 + C_{X_0}\right\}\right]$$

$$b_{12} = -K_{v\alpha} = \left[\frac{P_x}{mg}\sin\alpha_1 - \frac{P_z}{mg}\cos\alpha_1 - \frac{\rho V_{P_1}^2 S}{2mg}C_{X_\alpha}\right]$$

$$b_{13} = 1 = K_{v\theta_W}$$

$$b_{14} = 0$$

$$f_{11} = 0$$

$$f_{12} = K_{V\delta_T} = \frac{1}{mg}\left[\left(\frac{\partial P_x}{\partial \delta_T}\right)_1 \cos\alpha_1 + \left(\frac{\partial P_z}{\partial \delta_T}\right)_1 \sin\alpha_1\right]$$

(I-10) $\underline{b_{21}\bar{v}_p + b_{22}\bar{\alpha} + b_{23}\bar{\theta}_W + b_{24}\, q = f_{21}\bar{\delta}_T + f_{22}\bar{\delta}_e}$

where:

$$b_{21} = -K_{zV} = -\frac{\rho S}{m}\left[C_{z\alpha}\alpha_1 + C_{z_0} + C_{z_{\delta e}}\delta e_1\right]$$

$$b_{22} = D - K_{z\alpha} = D - \frac{1}{mV_{p_1}}\left[\frac{\rho V_{p_1}^2 S}{2} C_{z\alpha} - P_x\cos\alpha_1 - P_z\sin\alpha_1\right]$$

$$b_{23} = -K_{z\theta_W} = \frac{g}{V_{p_1}}\left[\sin\theta_{W_1}\right]$$

$$b_{24} = -\left[1 + K_{zq}\right] = -\left[1 + \frac{\rho S c}{4m} C_{z_Q}\right]$$

$$f_{21} = K_{z\delta_T} = \frac{1}{mV_{p_1}}\left[\left(\frac{\partial P_z}{\partial \delta_T}\right)_1 \cos\alpha_1 - \left(\frac{\partial P_x}{\partial \delta_T}\right)_1 \sin\alpha_1\right]$$

$$f_{22} = K_{z\delta_e} = \left[\frac{\rho V_p S}{2m} C_{z_{\delta e}}\right]$$

(I-11) $\underline{b_{31}\bar{v}_p + b_{32}\bar{\alpha} + b_{33}\bar{\theta}_W + b_{34}\, q = f_{31}\bar{\delta}_T + f_{32}\bar{\delta}_e}$

where:

$$b_{31} = -K_{qV} = -\left[\frac{\rho V_{p_1} S c}{I_{yy}}\left\{C_{M\alpha}\alpha_1 + C_{M_0} + C_{M_{\delta e}}\delta e_1\right\}\right]$$

$$b_{32} = -\left[K_{q\dot{\alpha}} D + K_{q\alpha}\right] = -\left[D\left\{\frac{\rho V_{p_1} S c^2}{4I_{yy}} C_{M\dot{\alpha}}\right\} + \frac{\rho V_{p_1}^2 S c}{2I_{yy}} C_{M\alpha}\right]$$

$$b_{33} = 0$$

$$b_{34} = D - K_{qq} = \left[D - \frac{\rho V_{p_1} S c^2}{4I_{yy}} C_{M_Q}\right]$$

$$f_{31} = 0$$

$$f_{32} = K_{q\delta_e} = \left[\frac{\rho V_{p_1}^2 S c}{2I_{yy}} C_{M_{\delta e}}\right]$$

The fourth equation is found from equation (I-7) and (I-8) as follows:

$\dot{\Theta}_W = Q - \dot{\alpha}$ which when linearized with $\dot{\Theta}_{W_1} = \dot{\alpha}_1 = Q_1 = 0$ provides:

$$\dot{\bar{\Theta}}_W + \dot{\bar{\alpha}} - q = 0$$

Rearranging and using operational form results in:

$$\text{(I-12)} \quad \underline{b_{41}\,\bar{v}_P + b_{42}\,\bar{\alpha} + b_{43}\,\bar{\Theta}_W + b_{44}\,q = f_{41}\,\bar{\delta}_T + f_{42}\,\bar{\delta}_e}$$

where:

$b_{41} = 0$

$b_{42} = D \equiv d/dt$

$b_{43} = D \equiv d/dt$

$b_{44} = -1$

$f_{41} = 0$

$f_{42} = 0$

B. Determination of the Elements of the Aircraft Matrix

The vector-matrix equation for controlling airspeed, $\bar{v}_P$, and the glide path angle, $\bar{\Theta}_W$, using power and elevator inputs, $\bar{\delta}_T$ and $\bar{\delta}_e$ is:

$$\text{(I-13)} \quad \begin{pmatrix} \bar{v}_P \\ \bar{\Theta}_W \end{pmatrix} = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{bmatrix} \begin{pmatrix} \bar{\delta}_T \\ \bar{\delta}_e \end{pmatrix} \quad \text{or} \quad \bar{y} = [\bar{A}]\,\bar{\delta}$$

When the relation above is expanded, the result is:

$$\text{(I-14)} \quad \bar{v}_P = \bar{a}_{11}\,\bar{\delta}_T + \bar{a}_{12}\,\bar{\delta}_e$$

$$\text{(I-15)} \quad \bar{\Theta}_W = \bar{a}_{21}\,\bar{\delta}_T + \bar{a}_{22}\,\bar{\delta}_e$$

The elements, $\bar{a}_{ij}$, in equations (I-14) and (I-15) are found by solving equations (I-9) through (I-12) simult-

aneously for $\bar{\nu}_p$ and $\bar{\theta}_w$. Using determinant methods yields:

$$\bar{\nu}_p = \frac{\begin{vmatrix} f_{11}\bar{\delta}_T & b_{12} & 1 & 0 \\ (f_{21}\bar{\delta}_T + f_{22}\bar{\delta}_e) & b_{22} & b_{23} & b_{24} \\ f_{32}\bar{\delta}_e & b_{32} & 0 & b_{34} \\ 0 & b_{42} & b_{43} & -1 \end{vmatrix}}{\Delta} \quad ; \quad \bar{\theta}_w = \frac{\begin{vmatrix} b_{11} & b_{12} & f_{11}\bar{\delta}_T & 0 \\ b_{21} & b_{22} & (f_{21}\bar{\delta}_T + f_{22}\bar{\delta}_e) & b_{24} \\ b_{31} & b_{32} & f_{32}\bar{\delta}_e & b_{34} \\ 0 & b_{42} & 0 & -1 \end{vmatrix}}{\Delta}$$

where

$$\Delta = \begin{vmatrix} b_{11} & b_{12} & 1 & 0 \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & 0 & b_{34} \\ 0 & b_{42} & b_{43} & -1 \end{vmatrix}$$

Expanding and simplifying the notation results in:

$$\text{(I-16)} \quad \bar{\nu}_p = \frac{\left[f_{11}(A) - f_{21}(B)\right]\bar{\delta}_T}{\Delta'} + \frac{\left[f_{32}(C) - f_{22}(B)\right]\bar{\delta}_e}{\Delta'}$$

$$\text{(I-17)} \quad \bar{\theta}_w = \frac{\left[f_{11}(E) - f_{21}(F)\right]\bar{\delta}_T}{\Delta'} + \frac{\left[f_{32}(G) - f_{22}(F)\right]\bar{\delta}_e}{\Delta'}$$

where:

$$(A) = \left[b_{23}b_{34}b_{42} + b_{24}b_{32}b_{43} + b_{32}b_{23} - b_{43}b_{34}b_{22}\right]$$

$$(B) = \left[b_{34}b_{42} + b_{32} - b_{34}b_{43}b_{12}\right]$$

$$(C) = \left[b_{24}b_{42} + b_{22} - b_{12}b_{23} - b_{43}b_{24}b_{12}\right]$$

$$(E) = \left[b_{24}b_{31}b_{42} + b_{22}b_{31} - b_{21}b_{32} - b_{42}b_{34}b_{21}\right]$$

$$(F) = \left[b_{31}b_{12} - b_{11}b_{32} - b_{42}b_{34}b_{11}\right]$$

$$(G) = \left[b_{12}b_{21} - b_{11}b_{22} - b_{42}b_{24}b_{11}\right]$$

$$\Delta' = \left[b_{11}(A) - b_{21}(B) + b_{31}(C)\right]$$

From equations (I-16) and (I-17) it is clear that the matrix elements are:

(I-18) $$\bar{a}_{11} = \frac{f_{11}(A) - f_{21}(B)}{\Delta'}$$

(I-19) $$\bar{a}_{12} = \frac{f_{32}(C) - f_{22}(B)}{\Delta'}$$

(I-20) $$\bar{a}_{21} = \frac{f_{11}(E) - f_{21}(F)}{\Delta'}$$

(I-21) $$\bar{a}_{22} = \frac{f_{32}(G) - f_{22}(F)}{\Delta'}$$

APPENDIX II

CALCULATION OF THE F-86 STABILITY DERIVITIVES AND DETERMINATION OF THE COEFFICIENTS, b_{ij} AND f_{ij}, OF APPENDIX I

A. Physical characteristics of the F-86.

The following data has been obtained from Ref. 6 .

$S = 288$ ft.^{2}

$W = 15{,}500$ lbs. (lightly loaded)

$b = 37.1$ ft.

$c = 8.09$ ft.

$I_{xx} = 8625$ slug-ft.^{2}

$I_{yy} = 29400$ slug-ft.^{2}

$I_{zx} = 2700$ slug-ft.^{2}

$I_{zz} = 35300$ slug-ft.^{2}

B. Assumed flight condition.

Altitude = sea level

Mach No. = .246

density = .002378 slug/ft^3

V_p = 163 knots = 275 ft/sec

$P_1 = R_1 = \Phi_1 = Q_1 = \dot{\imath}_1 = \dot{\alpha}_1 = \dot{\theta}_{w_1} = 0$

$\theta_w = -3^\circ = -.0523$ radians

The above data is for an F-86 proceeding down a three degree glide slope at 163 knots in a "clean" configuration.

C. Lift and Drag Coefficients at Trim; Stability Derivitives

1. $$C_{\mathcal{L}_{trim}} = \frac{2W\cos\theta_w}{\rho V_p^2 S} = .597$$

2. From the drag polar using $C_{\mathcal{L}} = .597$ as the entering argument: $C_{D_{trim}} = .0445$

Therefore, Drag $=$ 1152 lbs.

$$\alpha_1 = 9.5^{\circ}$$

3. Stability Derivatives

The following stability derivitives were obtained directly or indirectly from Ref. 6 .

$C_{\mathcal{L}_{\alpha}} = .065$ per degree $=$ 3.72 per radian

$C_{D_{\alpha}} = \dfrac{\partial C_D}{\partial C_{\mathcal{L}}} \dfrac{\partial C_{\mathcal{L}}}{\partial \alpha} =$ .361 per radian

$C_{M_{\alpha}} = \dfrac{\partial C_M}{\partial C_{\mathcal{L}}} \dfrac{\partial C_{\mathcal{L}}}{\partial \alpha} =$ -.433 per radian

$C_{m_q} = -5.6$ per radian

$C_{D_q} = \dfrac{\partial C_D}{\partial C_{\mathcal{L}}} \dfrac{\partial C_{\mathcal{L}}}{\partial Q} =$ .51 per radian

$C_{\mathcal{L}_q} = 5.1$ per radian

$C_{M_{\dot{\alpha}}} = -1.08$ per radian

$C_{\mathcal{L}_{\dot{\alpha}}} =$ .55 per radian

$C_{\mathcal{L}_{\delta_a}} = -.1775$ per radian

$C_{M_{\delta_s}} \equiv C_{M_{\delta_e}} =$ -.945 per radian

$C_{\mathcal{L}_{\delta_s}} \equiv C_{\mathcal{L}_{\delta_e}} \cong -\dfrac{C_{M_{\delta_s}}}{d_s/c} = .356$ per radian

4. Coe fficients of longitudinal equations

$$\left\{ \begin{array}{c} \text{The additional assumptions are :} \\ \delta e_1 = P_z = \tau_y = 0 \end{array} \right\}$$

a. $K_{vv} = \rho \dfrac{V_{p_1} S}{mg} \left\{ C_{X_{\alpha}} \alpha_1 + C_{X_0} \right\} = -\dfrac{2}{V_{p_1}} \dfrac{C_{D_{trim}}}{W/q_0 S} = -.00054 \ \dfrac{sec}{ft}$

b. $K_{V\alpha} = \frac{\rho V_{p_1}^2 S}{2mg} C_{x_\alpha} - \frac{P_x \sin\alpha_1}{mg} = \frac{q_0 S}{W}(-C_{D_\alpha}) - \frac{P_x \sin\alpha_1}{W}$

$K_{V\alpha} = -.6067$ per radian

c. $K_{V\theta_w} = 1$

d. $K_{V_{\delta_T}} = \frac{1}{mg}\left(\frac{\partial P_x}{\partial \delta_T}\right)_1 \cos\alpha_1 = .318$

e. $K_{zV} = \frac{\rho S}{m}\left[C_{z_\alpha}\alpha_1 + C_{z_0}\right] = \frac{\rho S}{m}\left[-C_{\mathcal{L}_{trim}}\right] = -.00085 \frac{1}{ft}$

f. $K_{z\alpha} = \frac{1}{mV_{p_1}}\left[\frac{1}{2}\rho V_{p_1}^2 S C_{z_\alpha} - P_x \cos\alpha_1\right] = \frac{g}{V_{p_1}}\left\{\frac{q_0 S}{W}(-C_{\mathcal{L}_\alpha}) - \frac{P_x \cos\alpha_1}{W}\right\}$

$K_{z\alpha} = -.709 \frac{1}{sec}$

g. $K_{z\theta_w} = -\frac{g}{V_{p_1}} \sin\theta_{w_1} = +.00613 \frac{1}{sec}$

h. $K_{zq} = \frac{\rho S C}{4m}\left[C_{z_q}\right] = \frac{\rho S C}{4m}\left(-C_{\mathcal{L}_q}\right) = -.0147 \frac{1}{rad.}$

i. $K_{z_{\delta_T}} = \left(-\frac{\partial P_x}{\partial \delta_T}\right)_1 \sin\alpha_1 \left(\frac{1}{mV_{p_1}}\right) = -.00624$

j. $K_{z\delta_e} = \frac{\rho V_{p_1} S}{2m} C_{z_{\delta_e}} = -\frac{\rho V_{p_1} S}{2m} C_{\mathcal{L}_{\delta_e}} = -.070 \frac{1}{sec}$

k. $K_{qV} = 0 \quad \{\text{since } T_y = 0\}$

l. $K_{q\dot{\alpha}} = \frac{\rho V_{p_1} S c^2}{4 I_{yy}} C_{m_{\dot{\alpha}}} = -.113 \frac{1}{sec}$

m. $K_{q\alpha} = \frac{\rho V_{p_1}^2 S c}{2 I_{yy}} C_{m_\alpha} = -3.09 \frac{1}{sec^2}$

n. $K_{qq} = \frac{\rho V_{p_1} S c^2}{4 I_{yy}} C_{m_q} = -.587 \frac{1}{sec}$

o. $K_{q\delta_e} = \frac{\rho V_{p_1}^2 S c}{2 I_{yy}} C_{m_{\delta_e}} = -6.74 \frac{1}{sec^2}$

D. Coefficients of Equations (I-9) through (I-12).

$b_{11} = D/g - K_{vv} = .0311D + .00654$

$b_{12} = -K_{v\alpha} = .6067$

$b_{13} = K_{v\theta_w} = 1.00$

$b_{14} = 0$

$f_{11} = K_{v\delta_T} = .318$

$b_{21} = -K_{zv} = .00085$

$b_{22} = D - K_{z\alpha} = D + .709$

$b_{23} = -K_{z\theta_w} = -.00613$

$b_{24} = -\left[1 + K_{zq}\right] = -.9853$

$f_{21} = K_{z\delta_T} = +.00624$

$f_{22} = K_{z\delta_e} = -.07$

$b_{31} = -K_{qv} = 0.00$

$b_{32} = -\left[DK_{q\dot{\alpha}} + K_{q\alpha}\right] = .113D + 3.09$

$b_{33} = 0$

$b_{34} = D - K_{qq} = D + .587$

$f_{31} = 0$

$f_{32} = K_{q\delta_e} = -6.74$

$b_{41} = 0$

$b_{42} = D \equiv d/dt$

$b_{43} = D \equiv d/dt$

$b_{44} = 0$

$f_{41} = 0$

$f_{42} = 0$

APPENDIX III

ELEMENTS OF THE SERVO AND TRANSDUCER MATRICES, T and S

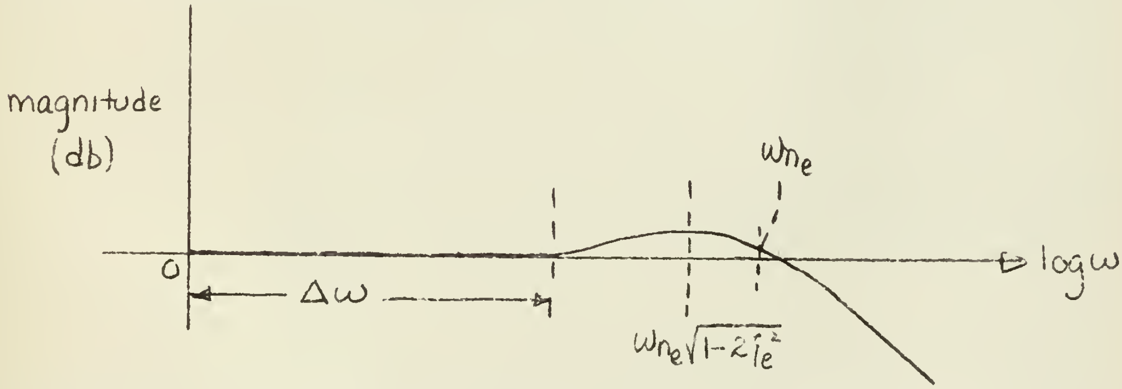
A. Servo Matrix Ele ments, s_{ii}.

1. The relation between elevator deflection, $\bar{\delta}$, and the signal to the elevator servo, δ, is defined as s_{11}.

A typical elevator servo system can be described as a second order system with natural frequency, ω_{n_e}, and damping ratio, ζ_e, The phase relation for such a system is given by:

$$(\text{III-1}) \qquad \phi = -\tan^{-1}\left\{\frac{2\zeta_e \frac{\omega}{\omega_{n_e}}}{1-\frac{\omega^2}{\omega_{n_e}^2}}\right\}$$

If $\zeta_e = .5$, the closed loop transfer function has a db versus log w characteristic as follows:



It follows that if the elevator servo is designed such that the natural frequency, ω_{n_e}, is larger than the expected range of system frequencies, $\Delta\omega$, or $\omega \ll \omega_{n_e}$ then equation (III-1) can be approximated by

$$(\text{III-2}) \qquad \phi = -\tan^{-1}\left\{\frac{2\zeta_e\omega}{\omega_{n_e}}\right\}$$

If the damping ratio is taken as .5, then the phase relation becomes:

$$(\text{III-3}) \qquad \phi = -\tan^{-1}\frac{\omega}{\omega_{n_e}} \equiv \tan^{-1}\tau\omega$$

where $\tau = \frac{1}{\omega_{n_e}}$

It may be shown that the phase relation of (III-3) is that phase relation for a simple first order system of the form:

(III-4) $$S(p) = \frac{1}{\frac{p}{\omega_n}+1} \equiv \frac{1}{\tau p+1}$$

In conclusion, if the damping ratio is chosen as .5 , and the natural frequency of the servo is taken as 25 rad/sec, then for the expected range of system frequencies the second order elevator servo system may be approximated by the first order system:

(III-5) $$S_{11}(p) = \frac{\bar{\delta}_e}{\delta_e} = \frac{1}{.04p+1}$$

2. The relation between effective throttle perturbation or thrust, $\bar{\delta}_T$, and the actual throttle perturbation, δ_T , is simply described by the following first order system:

(III-6) $$\frac{\bar{\delta}_T}{\delta_T} = \frac{1}{\tau p+1}$$

where τ = the time delay in obtaining the new engine RPM which is representative of the thrust.

The above relation is pointed out in Ref. 7 and is generally true for most turbojets. In addition, a typical time delay of two seconds, $\tau = 2.0$, was obtained from this reference. Thus:

(III-7) $$S_{22}(p) = \frac{\bar{\delta}_T}{\delta_T} = \frac{1}{2p+1}$$

B. Transducer Matrix Elements, t_{ii} .

1. The transducer that relates actual indicated aircraft

velocity, $\overline{v_p}$, to the measured aircraft indicated velocity is given by:

$$(III\text{-}8) \qquad t_{11}(p) = \frac{v_p}{\overline{v_p}} = \frac{1}{.2p+1}$$

Such a transfer function assumes no position errors in the airspeed indicator nor any dynamic errors in the instrument. The time delay, $\tau = .2$ sec , is due to the pressure transmission delay in the pitot system.

The basis of the foregoing statements is derived from a discussion of airspeed indicators and measuring devices contained in Ref. 8 .

2. The transducer that relates actual glide path angle, $\overline{\theta_\omega}$, to measured glide path angle, θ_ω , is defined as t_{22}. It is assumed that the angle θ and the angle of attack, α are obtained without error from a gyro and an angle of attack indicator. These two quantities are added together to produce $\overline{\theta_\omega}$. Therefore:

$$(III\text{-}9) \qquad t_{22}(p) = \frac{\theta_\omega}{\overline{\theta_\omega}} = 1$$

References

1. Boksenbom, Aaron S., and Hood, Richard: General Algebraic Method Applied to Control Analysis of Complex Engine Types. NACA Report 980, 1950.

2. Tsien, H. S.: Engineering Cybernetics, 1954; McGraw-Hill Book Co., Inc., New York.

3. A Synthesis Method for Multipole Control Systems, H. Freeman. Applications and Industry, March, 1957.

4. Stability and Physical Realizability Considerations in the Synthesis of Multipole Control Systems, H. Freeman. Applications and Industry, March 1958.

5. Howe, R. M.; Dynamics of Flight, 1956; University of Michigan, Ann Arbor, Michigan.

6. Revised F-86D Characteristics for Dynamic Stability and Autopilot Studies. North American Aviation, Inc., Report NA-50-107A, 1952.

7. Otto, Edward W., and Taylor, Burt L. III: Dynamics of a Turbojet Engine Considered as a Quasi-Static System. NACA Report 1011, 1951.

8. Coulthard, W. H.: Aircraft Instrument Design, 1952; Sir Isaac Pitman and Sons, Ltd.; London.